OPERATIONS RESEARCH

Expositions in Mathcad

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- *Optimizing methods*
- *Exame tasks*
- *Essays*
- *Programme solutions*

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Preface

The book **OPERATIONS RESEARCH – Expositions in Mathcad** is concise matter made during several years of teaching at the Tecnical Faculty “Mihailo Pupin“ in Zrenjanin. Subject matter of this book can be equaly interesting to students of many profiles of this faculty: informatics teachers, informatics engineers, technics and informatics teachers and to graduated engineers for management of technical systems and to the others. The book contains algorithms in operations research field that are mostly mathematically oriented. Its chapters treat:

- *Linear programming*
- *Transpotation problem*
- *Assignment problem of workplaces*
- *Nonlinear programming*
- *Dinamic programming*
- *Multicriteria optimization*
- *Heuristic research*
- *Markov chains*
- *Mass queueing*
- *Simulation modelling (Monte-Carlo)*
- *Game theory*
- *Inventory control*
- *Network planning*
- *Testing statistical hypotheses.*

After introductory informing about the history of the subject matter field, follow the problems exposed in the chapters of the cited sequence. Each chapter contains mathematical proceedings without detailed proofs, with accent on pragmatism and applicability of methods and proceedings of operations research. An accompanying CD completes the contents of the book and offers students sufficient and necessary matter through representative files, sorted according to chapters (folders).

The authors express their acknowledgement to PhD Milan Bojanović, director of the company CPS-CAD Professional Sys. from Belgrade and Mr Srđan Šerer, senior lecturer of English language, for useful suggestions at final editing of this textbook.
We especially express our thanks to the reviewers of this textbook, PhD Mirko Vujošević from the Faculty of Organizational Sciences from Belgrade, PhD Dragan Radojević from the Institute “Mihailo Pupin” in Belgrade and doc. Detelin Vasilev, PhD, from Higher school of transport “Todor Kableškov” in Sofia.

January, 2008 *Authors*

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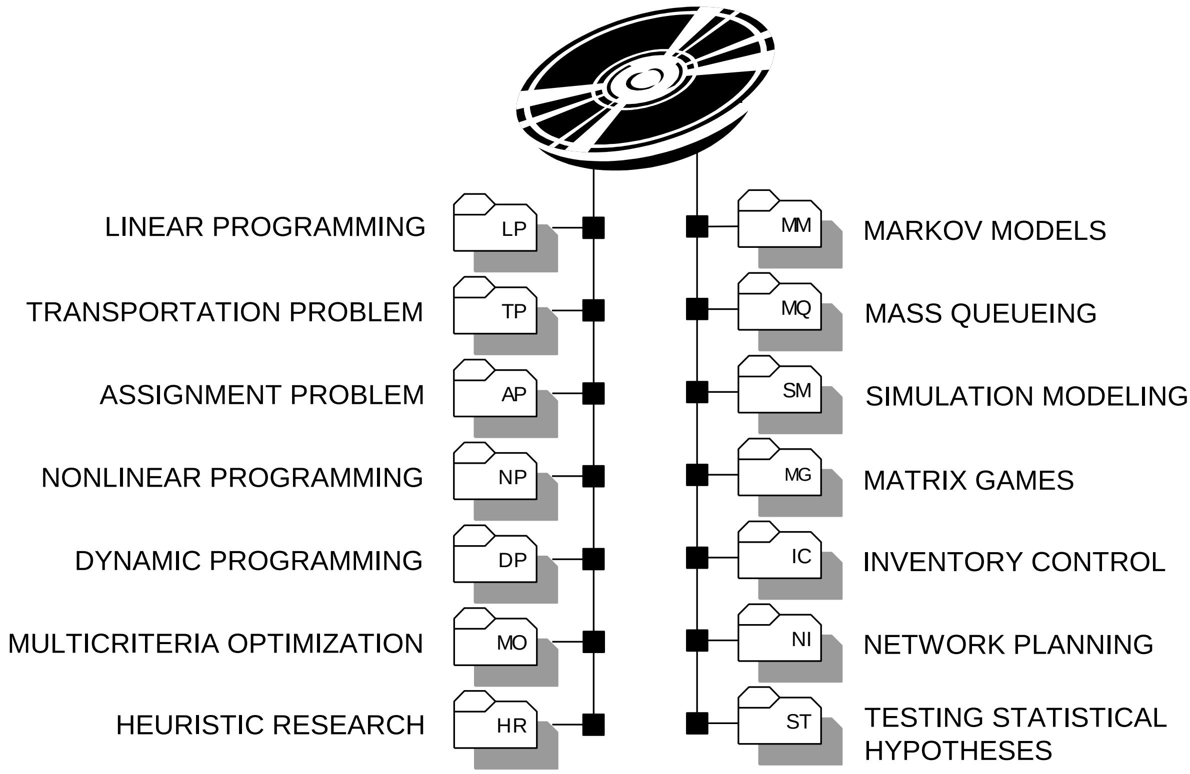

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OPERATIONS RESEARCH DEVELOPMENT

In computer mathematics, for models of often very complex structures, very quickly are being developed numerical methods of optimization. Their problems are studying exxtreme values of criteria functions with argument optimal values. We need to stress especially tasks of mathematical programming. Knowing their essentials, we can solve many tasks in production and business. Operations research and management science include also optimization tasks, as well appropriate methods elaborated for their solving. Stability of computing methods and algorithms comprises one of main trends in researching the application and classification of various mistakes with implementing these methods. In the beginning of the 20th century, in 1916, begins development of management science, founded by Frederick Taylor [2]. With this have been sloved several substantial matters, and they, predominantly, refer to resources management. In those years Harris has set fundamental mathematical model of optimal inventory level. The emergence of queueing theory referred to Danish mathematician A.K. Erlang (1909). Their contributions to the theory of optimal inventory have given, among others: Kendal, Lee, Hinchin and Polachek. Game theory is mathematical theory about decision making in competitor situations of opposite interests. The first to begin its developing, was Emile Borel in 1921, and fundamental contribution was given in 1928 by Von Neumann (1903-1957). Its first examples of modelling linear programming and transport problem were exposed in Kantorovich's papers in 1939, and in Neumann's work in 1936. Sudden development of this theory and praxis followed after the second world war. A transportation problem pioneer is also American mathematician Hitchcock from 1941. He was one of the first to formilate and solve one type of transportation assignments. There are also Vogel's papers dedicated to the approximate method for finding the initial transport solution. The papers of Ford, Falkerson, Charnes, Cuper and others. All these were preconditions to form a team of acientists on the eve of the second world war in Great Britain. They were from different fields but they were included into researches of multiple operations connected to coordination and arrangement of radar systems, military resources transport and the like. These operations researchers have used mathematical methods, and so they have created multidisciplinary science named Operations research. After the war its methods were very quickly implemented solving problems in production and business, medicine, industry and the like. Great importance had simplex, a linear programming method developed in 1947 by a famous American mathematician Danzing [9]. In the period 1951-1955 this nethod was modificated by Charens, Lemke and Danzig himself [9]. During fifties were being strongly sressed linear programming and statistical methods. During the same decade Neumann and Morgenstern have set a modern game theory. Since Neumann the term “games“ has been used as a scientific metaphor for communication among the expers for whom is substantial result of mutual strategies interaction of two or more parties that have conflict interests. The fifties were marked by the beginnings of non-linear programming by Kun and Tucker. This have coused eventful development of new methods and applying of operations research. The known numerical simulation method Monte Carlo was created in 1948, when appeared the paper of Metropolis and Ulman about random numbers. It has been used both for solving deterministic and stohastic tasks in many scientific spheres: reliability theory, meteorology, production, mass queueing, nuclear physics and the like. As logical continuation of traditional Gant's diagram aplying towards the end of fifties of last century, there was developed the set of methods that is called network planning technologies. These methods are established on results of algebra, graph theory, statistics and computing sciences. The first study with essential assumptions of this method was published by Valker and Kelly in 1958. The development of PERT method began in 1958. The

research was led by Fazar and mathematical bases of the method were established by Clark in 1958 [15]. Dynamic programming (DP) and algorithms of optimal management were set by American scientist Belman in 1952. He developed classical methodology for modelling and solving of one class of specially structured optimizing tasks connected with so-called multiphasal management processes. During 1960s great influence have had: network planning, linear programming, graph theory in optimization and discrete stohastic simulation. 1970s are characteristic with non-linear programming and global optimization, as well with computer methods based on numerical mathematics. Then have been significantly developed theoretical bases with new algorithms in uncertainty approach, when classical statistical and probability models have not been any more exclusively used. A significant technology was developed in the behinning of 70s. It was named PDM or “precedence diagramming method“. The algorithms of this method are built today in nearly all program packets for network planning. So, this highly elaborated technology is perhaps most of all others from the research field, applicable in praxis. In 80s the attention of operations research is directed to multicriteria optimiyation and decision making theory. Expert systems and the systems for decision making support enable introucting personal computers with appropriate program support. More recent methods, established on the theory of “fuzzy sets“ in the world of science were published by Zadeh in 1965. Nowdays there are numerous methods based on fuzzy principle: “As closer we watch real problem, its solution becomes more and more fuzzy“ (Zadeh), so that the theory of fuzzy sets has appropriately been applied in technical systems management [3], [25]. During 90s have been done significant penetrations in solving the problems of integer, mixed and multicriteria programming. Enrmous computer resources become mass available and enable efficient application of operations research method in daily real systems and processes. The development of new approaches for solving such problems, comprises on the basis of heuristics: genetic algorithms, neuron networks and the like. Gradually were being created the tools for solving the problems that will be applicated in new technological circumstances, internet environment and electronic business. Besides, we can say that an incomplete evolution in terminology has happened, so that nowdays, attention is paid to operational management, not only to their research.

1. LINEAR PROGRAMMING

1.1 Optimization of production program [LP1.mcd]

Example The factory produces two types of products P1 and P2. The production is being run on two machines. For the first product processing the operation times are: 4 hours/piece on the first machine and 10 hours/piece on the second machine. For processing the second product, these times are: 5 hours/piece on the first machine, and 6 hours/piece on the second machine. Effective capacities of the machines can be used to the atmost 750 and 850 hours respectively. The products appear at the market with minimal quantity of 40 pieces for P1 and 65 pieces for P2. Determine maximal production profit and optimal product quantities by the linear programming method, if the product unit prices are c1= 5500 mu/piece (mu=monetary unit) for P1 and c2 = 2800 mu/piece for P2.

Solution:

Criterion function (profit): $F(x) := 5500 \cdot x_0 + 2800 \cdot x_1$

In that case the vector of the products price is: $C := (5500 \quad 2800)$

Constraints in respect of production and market are:

production: $4 \cdot x_0 + 5 \cdot x_1 \le 750$ $\qquad 10 \cdot x_0 + 6 \cdot x_1 \le 850$

market: $x_0 \ge 40$ $\qquad x_1 \ge 65$

On account of equalization of relational operators of all four constraints, the last two non-equations can be defined as:

$$A := \begin{pmatrix} 4 & 5 \\ 10 & 6 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad B := \begin{pmatrix} 750 \\ 850 \\ -40 \\ -65 \end{pmatrix}$$

One initial value (it is supposed for the last variable): $x_1 := 50$

The block for solving the model of linear programming:

$$\text{Given} \qquad A \cdot x \le B \qquad x \ge 40$$

The optimal quantities of products P1 and P2 of the production program:

$Xr := \text{Maximize}(F, x)$ so the optimal values are: $Xr = \begin{pmatrix} 46 \\ 65 \end{pmatrix}$

or apart: $Xr_0 = 46$ $\qquad Xr_1 = 65$

The maximum company profit is: $D := |C \cdot Xr|$ $\qquad D = 435000$

Arithmetical an logical solution verification are: $A \cdot Xr = \begin{pmatrix} 509 \\ 850 \\ -46 \\ -65 \end{pmatrix}$ and $A \cdot Xr \ge 0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

Supplementory variables are: $Xd := A \cdot Xr - B$ $Xd = \begin{pmatrix} -241 \\ 0 \\ -6 \\ 0 \end{pmatrix}$

Note Alternative procedures for solving this problem are given at the files LP1_1.mcd.

1.2 Optimization of vitamin [LP2.mcd]

Example For the production of the vitamin tablets T1, T2 and T3 one factory that produces pharmaceuticals uses the raw materials S1, S2, S3 and S4. The number of vitamin tablets one can get out of one kilogram of raw material is presented in the next table [16].

T.1.1

S_j / T_i	*S1*	*S2*	*S3*	*S4*	*Need of tablets (pieces)*
T1	8	6	4	4	18000
T2	4	4	2	6	12000
T3	4	2	8	2	8000

During planned period it is necessary to produce 18000, 12000 and 8000 vitamin tablets T1, T2 and T3. One kilogram of raw material S1, S2, S3 and S4 one can buy at the market at the price of 16, 24 15 and 20 /mu/. If raw material supplying *Sj* has no constraints, one must determine optimal plan of raw materials aquisition with least total expenses.

Solution

Let x1, x2, x3 and x4 reprezents kilogram number of raw materials S1, S2, S3 and S4, then the model of raw materials acquisition has the form:

$$8 \cdot x_1 + 6 \cdot x_2 + 4 \cdot x_3 + 4 \cdot x_4 = 18000$$

$$4 \cdot x_1 + 4 \cdot x_2 + 2 \cdot x_3 + 6 \cdot x_4 = 12000$$

$$4 \cdot x_1 + 2 \cdot x_2 + 8 \cdot x_3 + 2 \cdot x_4 = 8000$$

The matrix of coefficiants and the constraints vector on the basis of the equation system is now:

$$A := \begin{pmatrix} 8 & 6 & 4 & 4 \\ 4 & 4 & 2 & 6 \\ 4 & 2 & 8 & 2 \end{pmatrix} \qquad B := \begin{pmatrix} 18000 \\ 12000 \\ 8000 \end{pmatrix} \qquad ORIGIN := 1$$

The criterion function is formed as: $F(x) := 16 \cdot x_1 + 24 \cdot x_2 + 15 \cdot x_3 + 20 \cdot x_4$

Where the price vector is: $C := (16 \;\; 24 \;\; 15 \;\; 20)$

One initial value (it is supposed for the last variable): $x_4 := 0$

The block for solving the model of linear programming:

$$\text{Given} \qquad A \cdot x = B \qquad x \geq 0$$

The optimal variants of cutting out bars: $X := \text{Minimize}(F, x)$

so, the optimal values are: $X^T = (1250 \;\; 1000 \;\; 0 \;\; 500)$

The optimal plan presents next values of raw materials quantity /kg/:

$$X_1 = 1250 \qquad X_2 = 1000 \qquad X_3 = 0 \qquad X_4 = 500$$

The minimum aggregate expenses of raw materials acquisition for the production of the planned number of vitamin tablets are /mu/:

$$T := |C \cdot X| \qquad T = 54000$$

The verification of solution: $A \cdot X = \begin{pmatrix} 18000 \\ 12000 \\ 8000 \end{pmatrix}$

1.2 Optimization of production program assortment/quantity [LP3.mcd]

Example Production program contains five products [32]: P1, P2, P3, P4 and P5, whose characteristics are given in the table:

- q_i – Quantities that can be planned for market /pieces/year/.
- d_j – Profit per product unit /dollar/piece/.

T.1.2

Products		*Units profil* d_j
P1	Helical spindle	14
P2	Unit for borders sharpening	10
P3	Grindstone spindle	7
P4	Universal partitioned apparatus	12
P5	Distance pads	9

In the work process analysis for the production of five planned products, besides others, have taken part five kinds of technological systems as like:

- π US – universal lathes CNC, with capacity of 50000 hours/year.
- π OC – process centres, with capacity of 80000 hours/year.
- π BK – coordinative drilling machines, with capacity of 40000 hours/year.
- π BR – grinder for flat grinding, with capacity of 50000 hours/year.
- π BN – grinder for thread grinding, with capacity of 20000 hours/year.

Here, time standard for products /hours/year/ and given sorts of technological systems is presented in the next table.

T.1.3

Technological systems	*Products*					*Time capacities /hours/*
	P1	*P2*	*P3*	*P4*	*P5*	
US	2	5	4	25	10	50000
OC	4	25	15	20	-	80000
BK	1	20	5	2	1	40000
BR	1	4	10	18	20	50000
BN	5	-	-	-	-	20000

Production program optimization will be realised in compliance with the above described LP model in which criterion function is given as the need for profit maximization.

System variable: $\mathrm{ORIGIN} := 1$

Criterion function (profit) is: $F(x) := 14 \cdot x_1 + 10 \cdot x_2 + 7 \cdot x_3 + 12 \cdot x_4 + 9 \cdot x_5$

The constraints defined only by thime capacities of technological systems are:

$$2 \cdot x_1 + 5 \cdot x_2 + 4 \cdot x_3 + 25 \cdot x_4 + 10 \cdot x_5 \le 50000$$

$$4 \cdot x_1 + 25 \cdot x_2 + 15 \cdot x_3 + 20 \cdot x_4 \le 80000$$

$$x_1 + 20 \cdot x_2 + 5 \cdot x_3 + 2 \cdot x_4 + x_5 \le 40000$$

$$x_1 + 4 \cdot x_2 + 10 \cdot x_3 + 18 \cdot x_4 + 20 \cdot x_5 \le 50000$$

$$5 \cdot x_1 \le 20000$$

It is necessary to formalize the previous LP model in the form of matrices and vectors as:

The vector of prices: $C := \begin{pmatrix} 14 & 10 & 7 & 12 & 9 \end{pmatrix}$

The matrix of coefficients (operation times) A and constraints vector (time capacities) B:

$$A := \begin{pmatrix} 2 & 5 & 4 & 25 & 10 \\ 4 & 25 & 15 & 20 & 0 \\ 1 & 20 & 5 & 2 & 1 \\ 1 & 4 & 10 & 18 & 20 \\ 5 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad B := \begin{pmatrix} 50000 \\ 80000 \\ 40000 \\ 50000 \\ 20000 \end{pmatrix}$$

The solution follows on the basis of the defined model through the matrix A and the vectors B and C, with the supposed initial value (as the last variable in non-equation system):

$$x_5 := 0$$

LP block for solving: Given

$$A \cdot x \le B \qquad x \ge 0$$

The optimal solution for products quantity: $Xr := \mathrm{Maximize}(F, x)$ $\quad Xr = \begin{pmatrix} 4000 \\ 1547.61 \\ 453.12 \\ 925.64 \\ 930.84 \end{pmatrix}$

or apart:

$$Xr_1 = 4000 \qquad Xr_2 = 1547.61 \qquad Xr_3 = 453.12 \qquad Xr_4 = 925.64 \qquad Xr_5 = 930.84$$

The maximal profit is: $D := |C \cdot Xr|$ $\quad D = 94133.24$

The verification of arithmetical and logical solution:

$$A \cdot Xr = \begin{pmatrix} 50000 \\ 80000 \\ 40000 \\ 50000 \\ 20000 \end{pmatrix} \qquad A \cdot Xr \le B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$ (True)

The additional variables are: $Xd := A \cdot Xr - B$ respectively: $Xd = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

1.4 Linear programing model with two variables [LP4.mcd]

Example For the given criterion function and the non-equation system with two unknowns, apply the method of optimum solving of unknown arguments and maximum of criterion funciton. On the basis of LP model form graphical and animation presentation of the solution.

Solution

Criterion function (max): $F(x) := 2\cdot x_1 + 2\cdot x_2$ $(\mathrm{ORIGIN} := 1)$

The system of linear constraints:

$$x_1 + 2\cdot x_2 \le 8 \qquad -x_1 + x_2 \le 1 \qquad -x_1 + 2\cdot x_2 \ge 0 \qquad \frac{4}{5}\cdot x_1 + 2\cdot x_2 \ge 3$$

On account of equalling relativ operators the last two non-equations can be written as:

$$x_1 - 2\cdot x_2 \le 0 \qquad -\frac{4}{5}\cdot x_1 - 2\cdot x_2 \le -3$$

On the basis of previous LP model follows the matrix of coefficients and the constraints vector:

$$A := \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & -2 \\ \frac{-4}{5} & -2 \end{pmatrix} \qquad B := \begin{pmatrix} 8 \\ 1 \\ 0 \\ -3 \end{pmatrix}$$

On the basis ov previous LP model, follows the matrix of coefficients and the constraints vector.

The initial value of one unknown (the last variable): $x_2 := 1$

Block for solving and LP system: Given $A\cdot x \le B$ $x \ge 0$

Optimal solution of LP model: $Xr := \mathrm{Maximize}(F, x)$ so: $Xr = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Maximal value of criterion function is: $F := |C\cdot Xr|$ $F = 12$

The solution verification:

$$A\cdot Xr = \begin{pmatrix} 8 \\ -2 \\ 0 \\ -7.2 \end{pmatrix} \qquad A\cdot Xr - B = \begin{pmatrix} 0 \\ -3 \\ 0 \\ -4.2 \end{pmatrix}$$

The graphical and animation solution of the linear problem:

$$\begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} 4 \\ 2 \end{pmatrix} \qquad Tr := 50\cdot FRAME$$

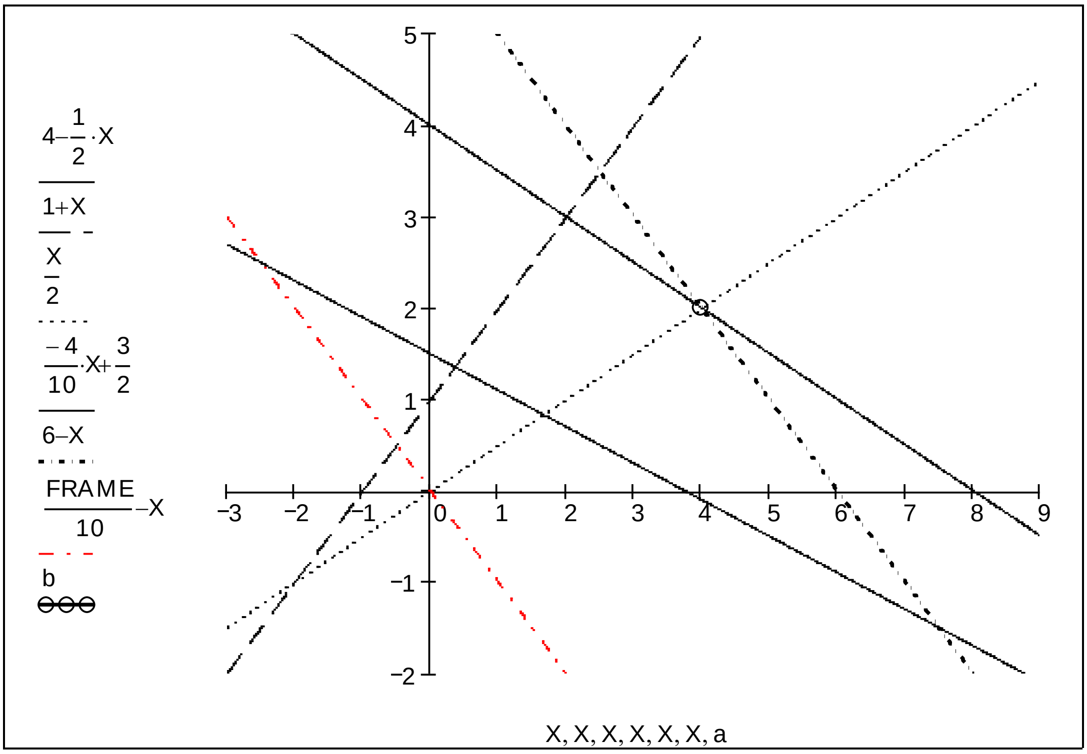


Fig. 1.1 *Graphical interpretation of the linear programming model with the presentation of the minimum (zero) and the maximum value of criterion function – profit*

1.5 Optimization of rocket number [LP5.mcd]

Example It is necessary to build a certain AAP (Anti Aircraft Protection) system. It is known that the enemy has 100 planes for the action from low heights, 150 planes for the action from medium heights and 100 planes for the action from great heights, but it is not known from which height will he act. It is possible to provide two types of rockets. The first type of rockets downs planes with possibilities: 3/4, 1/2 and 1/4, and the second type with possibilities: 1/4, 1/2 and 3/4, depending on their flying at low, medium of great heights. The first type costs 25, the second 50 /mu/piece/. How many and which rockets it is necessary to provide so that the expected number of downed planes will not be fewer than the number of planes that can act? Here is necessary the

expenses for acquisition reduce to the minimum possible measure, and to present the solution fraphically [20].

Solution

Criterion function (expenses): $ORIGIN := 1$ $F(x) := 25 \cdot x_1 + 50 \cdot x_2$

In that case the vector of cost price is: $C := (25 \quad 50)$

The constraints system concerning number of downed planes:

$$\frac{3}{4} \cdot x_1 + \frac{1}{4} \cdot x_2 \geq 100 \qquad \frac{1}{2} \cdot x_1 + \frac{1}{2} \cdot x_2 \geq 150 \qquad \frac{1}{4} \cdot x_1 + \frac{3}{4} \cdot x_2 \geq 100$$

The matrix of coefficeints and constraints vector on the basis of the non-equation system is now:

$$A := \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \qquad B := \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$$

One initial value (it is supposed for the last variable): $x_2 := 0$

The block for solving linear programming model: Given

$$A \cdot x \geq B \qquad x \geq 0$$

The optimal rocket quantities: $X := \text{Minimize}(F, x)$ $X = \begin{pmatrix} 250 \\ 50 \end{pmatrix}$

So, for construction an efficient AAP system, it is necessary to provide the next number of rockets:

$$X_1 = 250 \qquad X_2 = 50$$

The minimum value of criterion function is: $T := |C \cdot X|$ $T = 8750$

The verification of solution: $A \cdot X = \begin{pmatrix} 200 \\ 150 \\ 100 \end{pmatrix}$

The programming of graphic solution LP and scaling of the x-axis: $X := -100 .. 400$

$$f(X) := \begin{cases} -3 \cdot X + 400 & \text{if } 0 \le X \le 50 \\ -X + 300 & \text{if } 50 \le X \le 250 \\ \dfrac{-1}{3} \cdot X + \dfrac{400}{3} & \text{if } 250 \le X \le 400 \end{cases} \qquad \begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} 250 \\ 50 \end{pmatrix}$$

The bases for the animation realization of criterion function: $Tr := 50 \cdot FRAME$

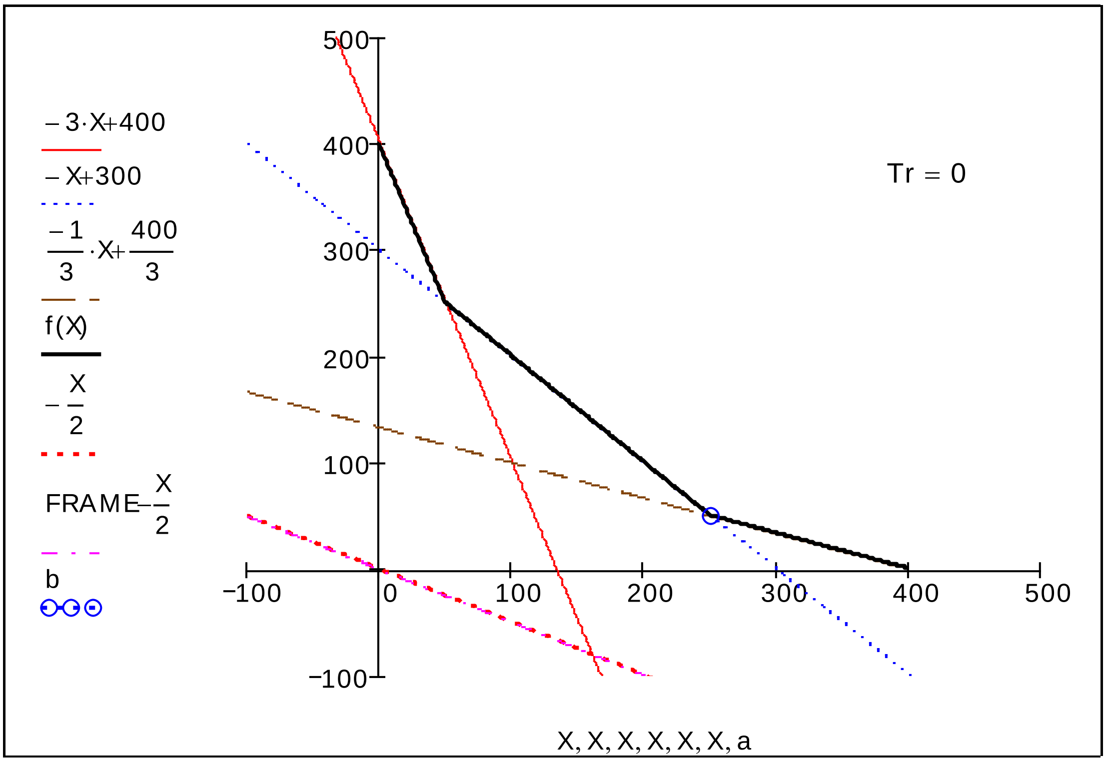


Fig. 1.2 *The graphical interpretation of linear programming model with the diagram of zero and the minimum value of criterion function – expenses*

1.6 Raw materials quantity optimization of chemical products [LP6.mcd]

Example Chemical industry produces three products P1, P2 and P3 in one separation process from two kinds of raw materials: S1 and S2. On the next table is presented how many tons of products is possible to get out of one ton of raw materials. The factory is obligated to deliver monthly to its partners at least 30, 25 and 20 tons respectively of separate products.

T.1.4

S_i \ P_j	*P1*	*P2*	*P3*
S1	0,03	0,125	0,4
S2	0,6	0,25	0,05
Quantity of products (tons)	30	25	20

a) How many tons of raw materials is necessary to supply the commercial department of chemical industry ant the price of 500 and 400 /mu/ respectively, so that the expanses are minimal.
b) Out of which product we can deliver more to the partners than the lower limit is.
c) Present in graphical and animation way the solution for the LP model.

Solution

Constraints system with regard to non-equations: $(\mathrm{ORIGIN} := 1)$

$$0.03 \cdot x_1 + 0.6 \cdot x_2 \geq 30 \qquad 0.125 \cdot x_1 + 0.25 \cdot x_2 \geq 25 \qquad 0.4 \cdot x_1 + 0.05 \cdot x_2 \geq 20$$

Criterion function (expenses): $F(x) := 500 \cdot x_1 + 400 \cdot x_2$

In that case the vector of products price is: $C := (500 \quad 400)$

Matrix of coefficients and constraints vector onthe basis of non-equations system is now:

$$A := \begin{pmatrix} 0.03 & 0.6 \\ 0.125 & 0.25 \\ 0.4 & 0.05 \end{pmatrix} \qquad B := \begin{pmatrix} 30 \\ 25 \\ 20 \end{pmatrix}$$

a) One initial value (it is supposed for the last variable): $x_2 := 0$

The block for solving LP model:

$$\text{Given} \qquad A \cdot x \geq B \qquad x \geq 0$$

The optimal quantities of P1 and P2 products of the production program:

$$Xr := \mathrm{Minimize}(F, x) \quad \text{or} \quad Xr = \begin{pmatrix} 40 \\ 80 \end{pmatrix}$$

Or separetly: $Xr_1 = 40$ $Xr_2 = 80$

The minimum expenses are: $T := |C \cdot Xr|$ $T = 52000$

Arithmetical and logical verification of the solution: $A \cdot Xr = \begin{pmatrix} 49.2 \\ 25 \\ 20 \end{pmatrix}$ $A \cdot Xr \geq 0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

b) Possibility for delivery of greater number of products

The additional variables are: $Xd := A \cdot Xr - B$ $Xd = \begin{pmatrix} 19.2 \\ 0 \\ 0 \end{pmatrix}$

Statement: From P1, the first product to the partners can be delivered for 19.2 tons more than the lower limit of 30 tons is.
Programming low limit of permissible solution:

$$f(X) := \begin{array}{|l} 400 - 8 \cdot X \text{ if } 0 \leq X \leq 40 \\ 100 - \dfrac{X}{2} \text{ if } 40 \leq X \leq \dfrac{1000}{9} \\ 50 - \dfrac{X}{20} \text{ if } \dfrac{1000}{9} \leq X \leq \infty \end{array}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} 40 \\ 80 \end{pmatrix}$$

$$Tr := 400 \cdot FRAME$$

Note For animation procedures are recommended next parameters: From: 0, To: 130, At: 10 frame/sec.

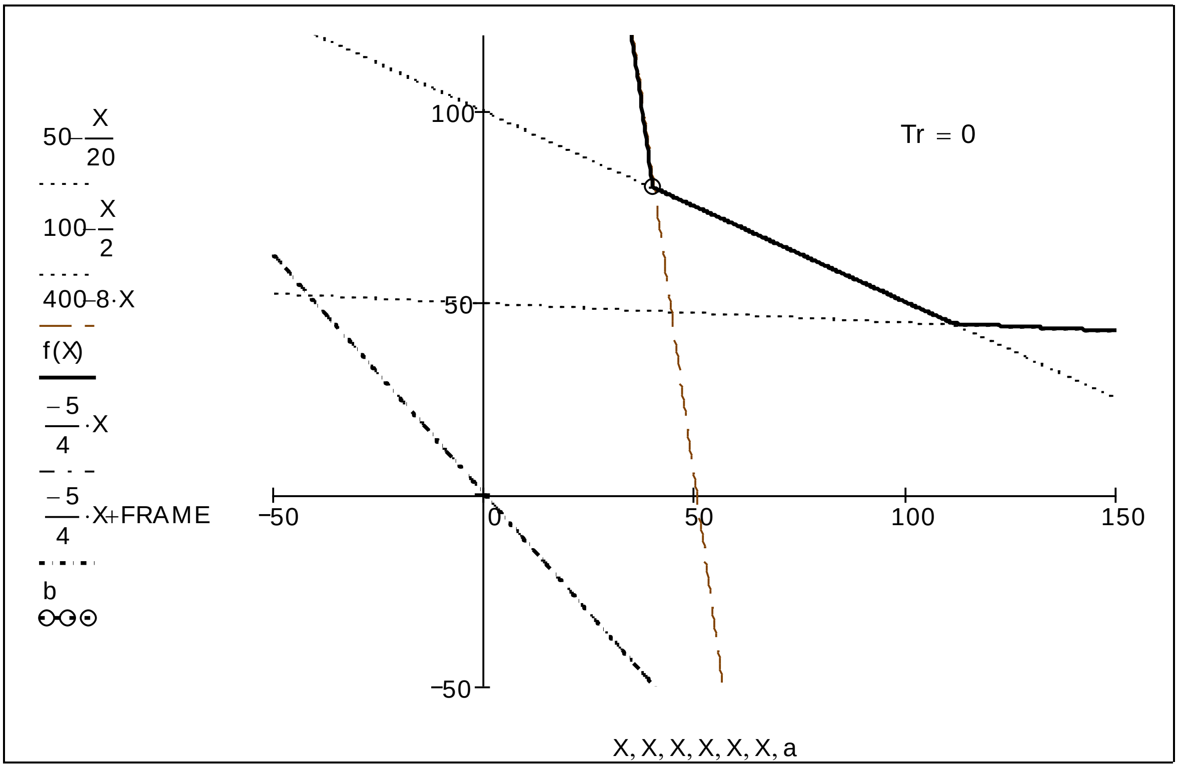


Fig. 1.3 *Graphical interpretation of linear programming model with the diagram of zero value of criterion function – expenses*

1.7 Dual model of linear programming [LP7.mcd]

Example For known criterion function and non-equation system, form primary and dual LP model. Find out optimal values of unknown arguments and minimal (for primary model) respecitvely maximal values (for dual model) of criterion function.

Primary LP model

Criterion function:

$$F(x) := 2 \cdot x_0 + 4 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3$$

Linear constraints system:

$$4 \cdot x_0 + 2 \cdot x_1 + 5 \cdot y_2 + 8 \cdot y_3 \geq 1800$$

$$25 \cdot x_0 + 75 \cdot x_1 + 2 \cdot y_2 + 40 \cdot y_3 \geq 2400$$

$$16 \cdot x_0 + 10 \cdot x_1 + 7 \cdot y_2 + 4 \cdot y_3 \geq 3000$$

Primary model is described by the next vectors and matrix of coefficients:

$$C := (2 \quad 4 \quad 2 \quad 3) \qquad A := \begin{pmatrix} 4 & 2 & 5 & 8 \\ 25 & 75 & 2 & 40 \\ 16 & 10 & 7 & 4 \end{pmatrix} \qquad B := \begin{pmatrix} 1800 \\ 2400 \\ 3000 \end{pmatrix}$$

The initial value: $x_3 := 1$

The block for solving previous LP model: Given

$$A \cdot x \geq B \qquad x \geq 0$$

The optimal solution of the LP model: $Xr := \text{Minimize}(F, x)$ $\quad Xr = \begin{pmatrix} 53.991 \\ 0 \\ 298.694 \\ 11.321 \end{pmatrix}$

Minimal value of criterion function is: $F := |C \cdot Xr|$ $\quad F = 739.33$

The solution verification: $A \cdot Xr = \begin{pmatrix} 1800 \\ 2400 \\ 3000 \end{pmatrix}$

π **Dual LP model**

Dual criterion function: $\Phi(y) := B^T \cdot y$

respectively: $\Phi(y) := 1800 \cdot y_0 + 2400 \cdot y_1 + 3000 \cdot y_2$

Linear constraints system:

$$4 \cdot y_0 + 25 \cdot y_1 + 16 \cdot y_2 \leq 2 \qquad 5 \cdot y_0 + 2 \cdot y_1 + 7 \cdot y_2 \leq 2$$

$$2 \cdot y_0 + 75 \cdot y_1 + 10 \cdot y_2 \leq 4 \qquad 8 \cdot y_0 + 40 \cdot y_1 + 4 \cdot y_2 \leq 3$$

The dual model is described by the next vectors and matrix of coefficients:

$$B^T = (1800 \quad 2400 \quad 3000) \qquad C^T = \begin{pmatrix} 2 \\ 4 \\ 2 \\ 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} 4 & 25 & 16 \\ 2 & 75 & 10 \\ 5 & 2 & 7 \\ 8 & 40 & 4 \end{pmatrix}$$

Introductory condition (initial value): $y_2 := 1$

Given $\quad A^T \cdot y \leq C^T \qquad y \geq 0$

Block for dual LP model solving:

Optimal solution of LP model: $Yr := \text{Maximize}(\Phi, y)$ $\quad Yr = \begin{pmatrix} 0.35015 \\ 0.00145 \\ 0.0352 \end{pmatrix}$

The maximum value of criterion function is: $\Phi := \left|B^T \cdot Yr\right|$ $\quad \Phi = 739.33$

Both values are equivalent, i.e.: $F = \Phi$, with solution verification: $A^T \cdot Yr = \begin{pmatrix} 2 \\ 1.161 \\ 2 \\ 3 \end{pmatrix}$

2. TRANSPORTATION PROBLEM

2.1 Transportation problem with minimal expenses [TP1.mcd]

Example For needs of three industrial centres, there is imported one sort of material from three different countries, according to offer and request [23]:

Offer: $P := \begin{pmatrix} 6000 \\ 8000 \\ 3000 \end{pmatrix}$ $\quad \sum P = 17000$ $\qquad$ Request: $T := \begin{pmatrix} 5000 \\ 2000 \\ 10000 \end{pmatrix}$ $\quad \sum T = 17000$

Transportation costs per import unit are given in the next table:

T.2.1

Centre / *Country*	*I*	*II*	*III*	*Offer*
A	2 / x_1=?	7 / x_2=?	5 / x_3=?	6000
B	3 / x_4=?	1 / x_5=?	4 / x_6=?	8000
C	5 / x_7=?	3 / x_8=?	7 / x_9=?	3000
Request	5000	2000	10000	Σ= 17000

Or through costs vector: $C := (2 \;\; 7 \;\; 5 \;\; 3 \;\; 1 \;\; 4 \;\; 5 \;\; 3 \;\; 7)$

a) Find initial transport program.
b) Define optimal transport program so that total transportation (import) costs would be minimal.

Solution

According to given elements, transport problem can be expressed by costs function:

$$F(x) := 2 \cdot x_0 + 7 \cdot x_1 + 5 \cdot x_2 + 3 \cdot x_3 + x_4 + 4 \cdot x_5 + 5 \cdot x_6 + 3 \cdot x_7 + 7 \cdot x_8$$

And coefficients matrix and constraints vector on the basis of equations system:

$$M := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad V := \mathrm{stack}(P, T) \qquad V = \begin{pmatrix} 6000 \\ 8000 \\ 3000 \\ 5000 \\ 2000 \\ 10000 \end{pmatrix}$$

As total of offers is equal to total of requests, this problem is of closed type:

$$\sum P = \sum T = 1 \quad \text{(true)}$$

One initial solution value of one (last) variable: $x_8 := 0$

Block for solving linear programming model: Given

Linear equations system: $M \cdot x = V \qquad x \ge 0$

Optimal quantities of transport quantities X:

$$X := \mathrm{Minimize}(F, x) \qquad X = \begin{pmatrix} 5000 \\ 0 \\ 1000 \\ 0 \\ 0 \\ 8000 \\ 0 \\ 2000 \\ 1000 \end{pmatrix}$$

Minimum system costs: $Tr := |C \cdot Xr| \qquad Tr = 60000$

Values of optimal transport:

$$\begin{pmatrix} X_0 & X_1 & X_2 \\ X_3 & X_4 & X_5 \\ X_6 & X_7 & X_8 \end{pmatrix} = \begin{pmatrix} 5000 & 0 & 1000 \\ 0 & 0 & 8000 \\ 0 & 2000 & 1000 \end{pmatrix}$$

LP solution verifikation:

$$M \cdot X = \begin{pmatrix} 6000 \\ 8000 \\ 3000 \\ 5000 \\ 2000 \\ 10000 \end{pmatrix}$$

2.2 Transport task with minimal expenses [TP2.mcd]

Example For needs of three business centres they import one sort of material from three various regions according to offer and request. Transport expenses per import unit are given in the next table.

T.2.2

Centre / *Region*	C_1 (17)	C_2 (21)	C_3 (41)	C_4 (14)	C_5 (24)
R_1 (25)	10 / x_1=?	8 / x_2=?	9 / x_3=?	6 / x_4=?	5 / x_5=?
R_2 (32)	5 / x_6=?	6 / x_7=?	4 / x_8=?	3 / x_9=?	8 / x_{10}=?
R_3 (40)	9 / x_{11}=?	7 / x_{12}=?	6 / x_{13}=?	4 / x_{14}=?	3 / x_{15}=?
R_4 (20)	14 / x_{16}=?	10 / x_{17}=?	8 / x_{18}=?	8 / x_{19}=?	8 / x_{20}=?

a) Find initial transport program.
b) Determine optimal transport program, so that total transport expenses would be minimal.

Prices vector is formed on the basis of Matrix tool palete, through direct input of data:

$$C := (10 \;\; 8 \;\; 9 \;\; 6 \;\; 5 \;\; 5 \;\; 6 \;\; 4 \;\; 3 \;\; 8 \;\; 9 \;\; 7 \;\; 6 \;\; 4 \;\; 3 \;\; 14 \;\; 10 \;\; 8 \;\; 8 \;\; 8)$$

According to given elements, transport problem can be expressed in the form of expenses function (criterion):

$$\mathrm{ORIGIN} := 1 \qquad n := 1..20 \qquad F(x) := \sum_n \left(C^T\right)_n \cdot x_n$$

With adequate matrix of coefficients and constraints vector in view of equations system. The matrix of coefficients for constraint per region capacity is:

$$P := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Matrix of coefficients for constraint per centre capacity is:

$$S := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We get complete matrix as a connected serie: $A := \text{stack}(P, S)$

$A =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
5	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
6	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
8	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
9	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1

The constraints vector in regions and centres is:

$$B := \begin{pmatrix} 25 \\ 32 \\ 40 \\ 20 \\ 17 \\ 21 \\ 41 \\ 14 \\ 24 \end{pmatrix}$$

One initial value (of the last member): $x_{20} := 0$

Block for solving linear programming model: Given

Linear equations system: $A \cdot x = B$ $\quad x \geq 0$

Optimal transport quantities: $X := \text{Minimize}(F, x)$

$$X^T = (0 \;\; 21 \;\; 0 \;\; 0 \;\; 4 \;\; 17 \;\; 0 \;\; 15 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 6 \;\; 14 \;\; 20 \;\; 0 \;\; 0 \;\; 20 \;\; 0 \;\; 0)$$

Minimal transport expenses are: $T := |C \cdot X|$ $\quad T = 645$

The values of optimal transport plan can be presented through matrix:

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ X_6 & X_7 & X_8 & X_9 & X_{10} \\ X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{16} & X_{17} & X_{18} & X_{19} & X_{20} \end{pmatrix} = \begin{pmatrix} 0 & 21 & 0 & 0 & 4 \\ 17 & 0 & 15 & 0 & 0 \\ 0 & 0 & 6 & 14 & 20 \\ 0 & 0 & 20 & 0 & 0 \end{pmatrix}$$

Solution verifikation: $(A\cdot X)^T = (25\ \ 32\ \ 40\ \ 20\ \ 17\ \ 21\ \ 41\ \ 14\ \ 24)$

or: $A\cdot X = B = 1$ (true)

2.3 Transport problem of profit maximum [TP3.mcd]

Example Electronic industry in its basic for organisations during a month shold produce 72000 product units. For this product ought to be found market in four consumer centres. Earnings per product unit depends on production place and placemnet place. Every plant produces 18000 product units. Market analysis has indicated that consumer centres P1, P2, P3 and P4 need 8000, 12000, 20000, 32000 units, respectively [20]. This analysis and transport conditions showed that earnings per product unit is as it is given in the table.

T.2.3

Centre / *Plant*	C1 (8000)	C2 (12000)	C3 (20000)	C4 (32000)
P1 (18000)	10 / y_1=?	50 / y_2=?	80 / y_3=?	30 / y_4=?
P2 (18000)	10 / y_5=?	60 / y_6=?	40 / y_7=?	50 / y_7=?
P3 (18000)	80 / y_8=?	90 / y_9=?	20 / y_{10}=?	70 / y_{11}=?
P4 (18000)	40 / y_{12}=?	60 / y_{13}=?	50 / y_{14}=?	20 / y_{15}=?

It is necessary to determine distribution of products in consumer centres, that will enable the greatest earnings and optimal transport values.

Solution

The vector of earnings value is brought in view of table data:

$$C := (40\ \ 50\ \ 80\ \ 30\ \ 10\ \ 60\ \ 40\ \ 50\ \ 80\ \ 90\ \ 20\ \ 70\ \ 40\ \ 60\ \ 50\ \ 20)$$

According to given elements, transport problem can be expressed in the formof profit function:

$$n := 0 \,..\, 15 \qquad\qquad D(y) := \sum_{n} \left(C^{T}\right)_{n} \cdot y_{n}$$

And appropriate constraints as linear equations:

$$y_0 + y_1 + y_3 + y_4 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 18000$$

$$0 + 0 + 0 + 0 + y_5 + y_6 + y_7 + y_8 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 18000$$

$$0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + y_9 + y_{10} + y_{11} + y_{12} + 0 + 0 + 0 + 0 = 18000$$

$$0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + y_{12} + y_{13} + y_{14} + y_{15} = 18000$$

$$y_0 + 0 + 0 + 0 + y_4 + 0 + 0 + 0 + y_8 + 0 + 0 + 0 + y_{12} + 0 + 0 + 0 = 8000$$

$$0 + y_1 + 0 + 0 + 0 + y_5 + 0 + 0 + 0 + y_9 + 0 + 0 + 0 + y_{13} + 0 + 0 = 12000$$

$$0 + 0 + y_2 + 0 + 0 + 0 + y_6 + 0 + 0 + 0 + y_{10} + 0 + 0 + 0 + y_{14} + 0 = 20000$$

$$0 + 0 + 0 + y_3 + 0 + 0 + 0 + y_7 + 0 + 0 + 0 + y_{11} + 0 + 0 + 0 + y_{15} = 32000$$

Advice Forming the previous system of constraints equations is not indispensably for solving transport problem, in contrast to bringing in the next expressions that refer to the vectors C and B and the matrix A.

The complete matrix of coefficients is formed by opening and filling in blank table. We have got the Input Table through the dialoyue of Componene Wizard from the menu Insert ❍ Componenet. The reason for table forming is that we can not form a matrix with more than 100 elements through the Insert Matrix procedure from the toolbar Matrix.

A :=

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
4	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
5	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0
7	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1

Constraints vector by regions and centres: $B := \begin{pmatrix} 18000 \\ 18000 \\ 18000 \\ 18000 \\ 8000 \\ 12000 \\ 20000 \\ 32000 \end{pmatrix}$

One initial value: $y_{15} := 2$

Block for solving linear programming: Given

Linear equations system: $A \cdot y = B$ $\quad y \geq 0$

Optimal quantities of transport quantities Y: $Y := \text{Maximize}(D, y)$

$$Y^T = (0\ \ 0\ \ 18000\ \ 0\ \ 0\ \ 0\ \ 0\ \ 18000\ \ 4000\ \ 0\ \ 0\ \ 14000\ \ 4000\ \ 12000\ \ 2000\ \ 0)$$

Maximum company profit is: $Pr := |C \cdot Y|$ $\quad Pr = 4620000$

The value of optimal transport plan, arranged as the matrix 4x4:

$$\begin{pmatrix} Y_0 & Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 & Y_7 \\ Y_8 & Y_9 & Y_{10} & Y_{11} \\ Y_{12} & Y_{13} & Y_{14} & Y_{15} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 18000 & 0 \\ 0 & 0 & 0 & 18000 \\ 4000 & 0 & 0 & 14000 \\ 4000 & 12000 & 2000 & 0 \end{pmatrix}$$

Solution verifikation: $(A \cdot Y)^T = (18000\ \ 18000\ \ 18000\ \ 18000\ \ 8000\ \ 12000\ \ 20000\ \ 32000)$

2.4 Transport problem of profit maximum [TP4.mcd]

Example A great ogricultural combinate has classified its land in view of its structure into six categories and also gave been determined both dimensions of land complexes that belong to single categories, as well yields of 7 main products in /mu/ per land registry acre of categorized land. The next table presents the obtained data [7]:

T.2.4

K_i \ P_j	*Wheat*	*Oat*	*Oet*	*Maize*	*Abfalfa*	*Potato*	*Turnip*	*Complex /a/*
K1	12	18	5	0	20	100	60	4000
K2	8	14	3	40	10	120	0	8000
K3	18	5	5	36	16	60	0	14000
K4	16	12	0	50	4	0	140	2000
K5	4	0	8	25	0	40	230	18000
K6	5	24	0	42	18	80	200	23000
Plan /a/	20000	16000	2000	24000	3000	1000	3000	69000 \ 69000

In accordance with sowing plan it is forseen that with single sulture will be sown respectively 20000, 16000, 2000, 24000, 3000, 1000 and 3000 land registry acres (a). Which sowing plan should be realized, if the aim is maximizing production quantity?

Solution

Vector of Prices is brought in in view of table data: ORIGIN := 1

C :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	12	18	5	0	20	100	60	8	14	3	40	10	120	0	18

According to given elements, transport problem can be expressed in form of profit function:

$$n := 1 .. 42 \qquad D(q) := \sum_{n} \left(C^T\right)_n \cdot q_n$$

And appropriate constraints in form of linear equations

Complete matrix of coefficienst is formed by opening and filling in blank of the table. We get Input Table thruogh the dialogue of Component Wizard from the menu Insert ❍ Component.

$A :=$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0
8	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0
9	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0
10	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
11	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0
12	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1
13	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0

Constraints vector by regions and centres: $B :=$

	1
1	4000
2	8000
3	14000
4	2000
5	18000
6	23000
7	20000
8	16000

One initial value (last): $q_{42} := 0$

Block for solving models of linear programming: Given

Linear equations system: $A \cdot q = B$ $\quad q \geq 0$

Optimal quantities of transport quantities: $Q := \mathrm{Minimize}(D, q)$

$Q^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0	0	0	4000	0	0	0	0	2000	0	1000	2000	0	3000	0

Maximum company profit is: $Pr := \left|C \cdot Q\right|$ $\quad Pr = 754000$

The value of optimal transport plan, arranged as matrix 6x7:

$$\begin{pmatrix} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 & Q_7 \\ Q_8 & Q_9 & Q_{10} & Q_{11} & Q_{12} & Q_{13} & Q_{14} \\ Q_{15} & Q_{16} & Q_{17} & Q_{18} & Q_{19} & Q_{20} & Q_{21} \\ Q_{22} & Q_{23} & Q_{24} & Q_{25} & Q_{26} & Q_{27} & Q_{28} \\ Q_{29} & Q_{30} & Q_{31} & Q_{32} & Q_{33} & Q_{34} & Q_{35} \\ Q_{36} & Q_{37} & Q_{38} & Q_{39} & Q_{40} & Q_{41} & Q_{42} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 4000 & 0 & 0 & 0 \\ 0 & 2000 & 0 & 1000 & 2000 & 0 & 3000 \\ 0 & 14000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 1000 & 0 \\ 0 & 0 & 0 & 18000 & 0 & 0 & 0 \\ 20000 & 0 & 2000 & 1000 & 0 & 0 & 0 \end{pmatrix}$$

Solution verifikation:

$(A \cdot Q)^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	4000	8000	14000	2000	18000	23000	20000	16000	2000	24000	3000	1000	3000

Note Look at the contents of TP4_1.mcd file, where the resulting matrix is program solved.

2.5 Transport problem of minimum expenses [TP5.mcd]

Example In the following table are given goods quantities necessary to be dispatched from forwarding stations FS, goods quantities wanted in admission stations AS, as well transport prices from every forwarding station to every admission station.

T.2.5

PS / *OC*	*B1*	*B2*	*B3*	*B4*	*B5*	*Goods quantity /piece/*
A1	5	12	1	4	13	36
A2	7	8	14	6	5	23
A3	15	4	2	7	9	29
A4	6	11	5	16	3	12
Goods quantity /piece/	13	24	15	21	27	Σ=100

Which transport plan to be realized if the objective is minimizing of total transport expenses?

Solution

Vector of prices is brought in in view of table data: (ORIGIN := 1)

C :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	5	12	1	4	13	7	8	14	6	5	15	4	2	7	9	6

According to given elements, transport problem can be expressed in the form of profit function:

$$n := 1..20 \qquad T(x) := \sum_n \left(C^T\right)_n \cdot x_n$$

And adequate constraints in form of linear equations.

Complete matrix of coefficients is formed by opening and filling in blank table. We get Input Table through the dialogue Component Wizard from the menu Insert ❍ Component.

A :=

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	1	1	1	1	0	0	0
3	0	0	0	0	0	0	0	0	0	0	1	1	1
4	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	0	0	0	0	1	0	0	0	0	1	0	0
6	0	1	0	0	0	0	1	0	0	0	0	1	0
7	0	0	1	0	0	0	0	1	0	0	0	0	1
8	0	0	0	1	0	0	0	0	1	0	0	0	0
9	0	0	0	0	1	0	0	0	0	1	0	0	0

Constraint vector by regions and centres:

B :=

	1
1	36
2	23
3	29
4	12
5	13
6	24
7	15
8	21
9	27

One initial value (the last one): $x_{20} := 0$

Block for solving linear programming model: Given

Linear equations system: $A \cdot x = B \qquad x \geq 0$

Optimal quantities of transport quantities q: $q := \mathrm{Minimize}(T, x)$

$$q^T = (5\ \ 0\ \ 10\ \ 21\ \ 0\ \ 8\ \ 0\ \ 0\ \ 0\ \ 15\ \ 0\ \ 24\ \ 5\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 12)$$

Minimum expenses are: $Tr := |C \cdot q| \qquad Tr = 392$

The values of optimal transport plan, arranged as 4x5 matrix:

$$\begin{pmatrix} q_1 & q_2 & q_3 & q_4 & q_5 \\ q_6 & q_7 & q_8 & q_9 & q_{10} \\ q_{11} & q_{12} & q_{13} & q_{14} & q_{15} \\ q_{16} & q_{17} & q_{18} & q_{19} & q_{20} \end{pmatrix} = \begin{pmatrix} 5 & 0 & 10 & 21 & 0 \\ 8 & 0 & 0 & 0 & 15 \\ 0 & 24 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 12 \end{pmatrix}$$

Solution verifikation: $(A \cdot q)^T = (36 \;\; 23 \;\; 29 \;\; 12 \;\; 13 \;\; 24 \;\; 15 \;\; 21 \;\; 27)$

2.6 Transport problem of minimum of expenses [TP6.mcd]

Example For needs of seven business centres they import a sort of material from eight various countries according to offer and request (T.2.6). Transport expenses per import unit are given in the next table.

T.2.6

Country \ *Centre*	*P1*	*P2*	*P3*	*P4*	*P5*	*P6*	*P7*	*Offer*
S1	1 / x_1=?	3 / x_2=?	5 / x_3=?	7 / x_4=?	9 / x_5=?	11 / x_6=?	8 / x_7=?	300
S2	14 / x_8=?	12 / x_9=?	10 / x_{10}=?	6 / x_{11}=?	8 / x_{12}=?	6 / x_{13}=?	3 / x_{14}=?	290
S3	7 / x_{15}=?	9 / x_{16}=?	13 / x_{17}=?	17 / x_{18}=?	21 / x_{19}=?	24 / x_{20}=?	29 / x_{21}=?	280
S4	35 / x_{22}=?	22 / x_{23}=?	4 / x_{24}=?	26 / x_{25}=?	23 / x_{26}=?	20 / x_{27}=?	20 / x_{28}=?	270
S5	7 / x_{29}=?	10 / x_{30}=?	13 / x_{31}=?	16 / x_{32}=?	12 / x_{33}=?	8 / x_{34}=?	5 / x_{35}=?	260
S6	25 / x_{36}=?	20 / x_{37}=?	15 / x_{38}=?	5 / x_{39}=?	11 / x_{40}=?	17 / x_{41}=?	23 / x_{42}=?	250
S7	10 / x_{43}=?	15 / x_{44}=?	9 / x_{45}=?	16 / x_{46}=?	12 / x_{47}=?	17 / x_{48}=?	10 / x_{49}=?	240
S8	2 / x_{50}=?	5 / x_{51}=?	8 / x_{52}=?	8 / x_{53}=?	3 / x_{54}=?	15 / x_{55}=?	21 / x_{56}=?	230
Request	170	120	190	330	490	390	430	Σ= 2120

a) Find initial transport p rogram.
b) Determine optimal transport program so that total transport expenses are minimal.

Vector of prices is formed in view of Matrix toolbar through direct bringing in data:

C :=

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	1	3	5	7	9	11	8	14	12	10	6	8	6	3	7

According to biven elements, transport problem can be expressed in the form of expenses function (criterion):

$\mathrm{ORIGIN} := 1 \qquad n := 1 .. 56 \qquad F(x) := \sum_{n} \left(C^{T}\right)_{n} \cdot x_{n}$

With appropriate matrix of coefficients and constraints vector in view of system of equations. The matrix of coefficients for constraint for region capacities is:

A :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	1	1	1	1	1	1	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Number of elements in matrix: $s := \left(\mathrm{last}\left(A^{\langle 1\rangle}\right) + 1\right) \cdot \left[\mathrm{last}\left[\left(A^{T}\right)^{\langle 1\rangle}\right] + 1\right] \qquad s = 912$

Constraints vector for regions and centres is: B :=

	1
1	300
2	290
3	280
4	270
5	260
6	250
7	240
8	230

One initial value (the last one): $x_{56} := 0$

Block for solving the model of linearn programming: Given

System of linear equations: $A \cdot x = B \qquad x \geq 0$

Optimal transport quantities: $X := \mathrm{Minimize}(F, x)$

$X^{T} =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0	10	0	80	210	0	0	0	0	0	0	0	290	0

Minimum transport expenses are: $T := \left|C \cdot X\right| \qquad T = 14560$

Values of optimal transport plan can be pressented through matrix:

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 \\ X_8 & X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} \\ X_{15} & X_{16} & X_{17} & X_{18} & X_{19} & X_{20} & X_{21} \\ X_{22} & X_{23} & X_{24} & X_{25} & X_{26} & X_{27} & X_{28} \\ X_{29} & X_{30} & X_{31} & X_{32} & X_{33} & X_{34} & X_{35} \\ X_{36} & X_{37} & X_{38} & X_{39} & X_{40} & X_{41} & X_{42} \\ X_{43} & X_{44} & X_{45} & X_{46} & X_{47} & X_{48} & X_{49} \\ X_{50} & X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \end{pmatrix} = \begin{pmatrix} 0 & 10 & 0 & 80 & 210 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 290 & 0 \\ 170 & 110 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 190 & 0 & 0 & 80 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20 & 240 \\ 0 & 0 & 0 & 250 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 50 & 0 & 190 \\ 0 & 0 & 0 & 0 & 230 & 0 & 0 \end{pmatrix}$$

Solution verification: $(A \cdot X)^T =$

	1	2	3	4	5	6	7	8	9	10
1	300	290	280	270	260	250	240	230	170	120

2.7 Transport problem of minimum expenses [TP7.mcd]

Example Three mills (M1, M2 and M3) supplying with flour three bakery companies (P1, P2 and P3). In the next month the mills can produce 750, 400 and 350 tons of flour, respectively. In the same period the bakery companies shoul get: 75, 40 and35 tons of flour respectivey. Transport expenses per ton of flour from mill to baker's companies are: from the first mill 70, 30 and 60 /mu/ per Pj company respectively, from the second mill 40, 80 and 20 /mu/ per Pj company respectively and from the third mill 10, 50 and 90 /mu/ per Pj company respectively (T.2.7).

T.2.7

Bakeries / *Mills*	*P1*	*P2*	*P3*	*Capacities of bakeries /t/*
M1	70 / x_1=?	30 / x_2=?	60 / x_3=?	a_1= 75
M2	40 / x_4=?	80 / x_5=?	20 / x_6=?	a_2= 40
M3	10 / x_7=?	50 / x_8=?	90 / x_9=?	a_3= 35
Capacities of mills /t/	b_1= 20	b_2= 45	b_3= 30	Σ= 150 / Σ= 95

Find optimal transport plan, to which will suit minimal expenses.

Solution

Mathematical model is formed in order to determine open transport problem where we ought to determine value of non-negative variables x_{ij}. Here are initial parameters:

Number of rows: $m := 3$ number of columns: $n := 3$ $(\mathrm{ORIGIN} := 1)$

Index values: $i := 1..3$ $j := 1..3$

Capacities in vector form: $a := \begin{pmatrix} 75 \\ 40 \\ 35 \end{pmatrix}$ $b := \begin{pmatrix} 20 \\ 45 \\ 30 \end{pmatrix}$

Sum of capacities: $\sum a = 150$ $\sum b = 95$ $\sum a \neq \sum b$

And it is open transport problem. Becouse request is greater than offer, we bring in a fictitions bakery P4 with the capacity:

$$b_4 := \sum a - \sum b \qquad \text{follows that:} \qquad b_4 = 0$$

Capacities in expanded vector are: $a := \begin{pmatrix} 75 \\ 40 \\ 35 \end{pmatrix}$ $b := \begin{pmatrix} 20 \\ 45 \\ 30 \\ b_4 \end{pmatrix}$

T.2.8

Bakeries / *Mills*	*P1*	*P2*	*P3*	*P4*	*Capacities of bakeries /t/*
M1	70 / x_1=?	30 / x_2=?	60 / x_3=?	0 / x_4=?	a_1=75
M2	40 / x_4=?	80 / x_5=?	20 / x_6=?	0 / x_8=?	a_2=40
M3	10 / x_7=?	50 / x_8=?	90 / x_9=?	0 / x_{12}=?	a_3=35
Capacities of mills /t/	b_1= 20	b_2= 45	b_3= 30	b_4= 55	Σ= 150 / Σ= 95

Vector of expenses value is formed through direct data input from the expanded table T.2.8:

$$C := (70 \quad 30 \quad 60 \quad 0 \quad 40 \quad 80 \quad 20 \quad 0 \quad 10 \quad 50 \quad 90 \quad 0)$$

According to the given elements, transport problem can be expressed in form of expenses function (criterion):

$$k := 1..(n+1)\cdot m \qquad F(x) := \sum_k \left(C^T\right)_k \cdot x_k$$

With adequate matrix of coefficients and constraints vector in view of ewuations system. The matrix of coefficients for constraint for all capacities is:

A :=

	1	2	3	4	5	6	7	8	9	10	11	12
2	0	0	0	0	1	1	1	1	0	0	0	0
3	0	0	0	0	0	0	0	0	1	1	1	1
4	1	0	0	0	1	0	0	0	1	0	0	0
5	0	1	0	0	0	1	0	0	0	1	0	0
6	0	0	1	0	0	0	1	0	0	0	1	0

Constraints vector for capacities is: $B := \text{stack}(a, b)$ $B = \begin{pmatrix} 75 \\ 40 \\ 35 \\ 20 \\ 45 \\ 30 \\ 0 \end{pmatrix}$

One initial value (the last one): $x_{12} := 0$

Block for solving linear programming model: Given

Linear equations system: $A \cdot x = B$ $x \geq 0$

Optimal transport quantities X: $X := \text{Minimize}(F, x)$

$$X^T = (0 \quad 45 \quad 0 \quad 30 \quad 0 \quad 0 \quad 30 \quad 10 \quad 20 \quad 0 \quad 0 \quad 15)$$

Minimum transport expenses are: $T := |C \cdot X|$ $T = 2150$

Values of optimal transport plan can be presented through matrix:

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ X_5 & X_6 & X_7 & X_8 \\ X_9 & X_{10} & X_{11} & X_{12} \end{pmatrix} = \begin{pmatrix} 0 & 45 & 0 & 30 \\ 0 & 0 & 30 & 10 \\ 20 & 0 & 0 & 15 \end{pmatrix}$$

Arithmetical and logical solution verification: $A \cdot X = \begin{pmatrix} 75 \\ 40 \\ 35 \\ 20 \\ 45 \\ 30 \\ 55 \end{pmatrix}$ and $A \cdot X = B = 1$ (true).

3. ASSIGNMENTS PROBLEMS

3.1 Employees assignment [AP1a.mcd]

Example Production system in which are being manufactured final products from wood has advertised an vacancy for 5 macnihe carpenters for 5 working places (planer, milling machine, lathe, loring machine, grinder). After analyzing registration forms for the competition, all appointed conditions have been satisfied by five candidates. In order to space workers evenly to working places, there was organized probation [30]. Each candidate has got to process 100 same pieces on every machine and then was found a number of good pieces that was spoilage (T.3.1). The assignment was to dispose workers on working places, and that will provide minimal total spoilage.

T.3.1

Workers	*Number of points*				
	P1	P2	P3	P4	P5
R1	3	21	12	6	10
R2	8	23	2	5	5
R3	33	14	13	10	7
R4	14	21	19	11	11
R5	9	16	10	15	13

Solution

Values vector of non-standard (bad) products: ORIGIN := 1

Criterion function, as function of minimal spoilages: n := 5 m := 5

C :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	3	21	12	6	10	8	23	2	5	5	33	14	13	10	7	14	21	19	11	11	9	16	10	15	13

Criterion function, as function of minimal spoilages:

$$j := 1 .. n \cdot m \qquad D(x) := \sum_{j=1}^{n \cdot m} \left(C^T\right)_j \cdot x_j$$

Complete matrix of constraints is formed by opening and filling in blank table. Input table has been got through Component wizard dialoge from Insert ❍ Component menu.

Table A size: $(n + m) \cdot n \cdot m = 250$

Vector B size: $m + n = 10$

A :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
6	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
7	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
8	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
9	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
10	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1

Constraints vector that defines possibility that only one worker can work at one machine and that only one work can be assigned to one worker:

$$B := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

One initial value (the last one): $x_{25} := 0$

Block for solving linear programming model: Given

Linear equations system: $A \cdot x = B$ $\quad x \geq 0$

Optimal arrangement of work: $X := \mathrm{Minimize}(D, x)$

$X^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	1	0	0	0

Optimal plan values of work arrangement, as the matrix 5x5:

Minimal total numeber of spoilages: $S := |C \cdot X|$ $\quad S = 39$

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ X_6 & X_7 & X_8 & X_9 & X_{10} \\ X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{16} & X_{17} & X_{18} & X_{19} & X_{20} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Solution verification: $(A \cdot X)^T = (1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)$

Conclusion On the basis of optimal solution we state that the first job should be done to the first worker, the second to the fifth, the fhird to the second, the fourth to the fourth and the fifth to dhe fhird. Such arrangement of jobs will show the minimum number of spoiled products of 39 pieces.

3.2 Arrangement of jobs [AP2a.mcd]

Example In production system where are made final wood products, aught to be disposed 5 workers at five working places. While the competition was finished, all assigned conditions satisfied 5 candidates. In order to space workers at working places, there was organized test for their vocational capabilities in view of several criterions [25]. Vocational capability was expressed by summary number of won points, given in T.3.2.

T.3.2

Workers	*Number of points*				
	P1	P2	P3	P4	P5
R1	10	20	8	18	12
R2	20	30	10	15	17
R3	25	20	20	30	16
R4	18	15	15	20	22
R5	10	20	30	30	20

The assignment is to dispose workers at working places which will provide that total efficiency, expressed through number of points, will be maximal.

Solution

Vector of points value: $(ORIGIN := 1)$

C :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	10	20	8	18	12	20	30	10	15	17	25	20	20	30	16	18	15	15	20	22	10	20	30	30	20

Criterion function, as function of maximal points: $n := 5$ $m := 5$

$$j := 1..25 \qquad\qquad D(x) := \sum_j \left(C^T\right)_j \cdot x_j$$

Complete matrix of constraints is formed by opening and filling in blank table. Input table has been got through Component wizard dialoge from Insert – Component menu.

The size of A table: $(n+m)\cdot n\cdot m = 250$ vector B size: $m+n = 10$

A :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
6	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
7	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
8	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
9	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
10	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1

The constraints vector that defines the possibility that only one worker can work at only one machine, and that only one job can be given to one worker:

$$B := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Block for solving linear programming model: Given

Linear equations system: $A\cdot x = B \qquad x \geq 0$

Optimal works disposition: $X := \mathrm{Maximize}(D, x)$

$X^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	0	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1	0	0

Maximum value of criterion function is: $S := |C \cdot X|$ $S = 125$

The value of optimal arrangement of jobs, arranged as 5x5 matrix:

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ X_6 & X_7 & X_8 & X_9 & X_{10} \\ X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{16} & X_{17} & X_{18} & X_{19} & X_{20} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Solution verifikation: $(A \cdot X)^T = (1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)$

Conclusion In view of optimal solution it is stated that the fourth job ought to be given to the first worker, the second one to the second worker, the third to the first, the fourth to the fifth and the fifth to the fourth. Such job disposition will give the greated value of the criterion function, 125 points.

3.3 Job disposition [AP3a_2.mcd]

Example Working organisation ought to open foru new working places and for them was advertised vacancy [30]. There were five candidates in narrower selection. Their vocational abilities for doing jobs were tested. The number of won points was given in the table T.3.3.

T.3.3

Work places / *Machines*	M1	M2	M3	M4
R1	5	6	5	1
R2	4	6	4	1
R3	8	6	7	6
R4	2	4	4	4
R5	6	10	9	4

How ought to dispose workers at working places, so that whole efficency would be maximum? Which worker will not be admitted?

Solution

Previosly defined assignment problem has been opened. After adding one fictitious working place, we get the table T.4.4, and iti is condititon for forming square matrix 5x5 and apparently closed assignment problem.

T.3.4

Work places / *Machines*	M1	M2	M3	M4	M5
R1	5	6	5	1	0
R2	4	6	4	1	0
R3	8	6	7	6	0
R4	2	4	4	4	0
R5	6	10	9	4	0

Vector of points value: $(\mathrm{ORIGIN} := 1)$

$C :=$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	5	6	5	1	0	4	6	4	1	0	8	6	7	6	0	2	4	4	4	0	6	10	9	4	0

Criterion function, as function of maximal points: $n := 5$ $m := 5$

$$j := 1 .. n \cdot m \qquad D(x) := \sum_j \left(C^T\right)_j \cdot x_j$$

Complete matrix of constraints is formed by opening and filling in blank table. Input table has been got through Component Wizard dialoge from Insert ❍ Component menu.

Table A size: $(n + m) \cdot n \cdot m = 250$ vector B size: $m + n = 10$

$A :=$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1
6	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0
7	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0
8	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
9	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
10	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1

Constraint vector that defines the possibility that only one worker can work at one machine and that only one job can be given to one worker:

$$B := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

One initial value (the last): $x_{25} := 0$

Block for solving linear programming model: Given

System of linear equations: $A \cdot x = B$ $\quad x \geq 0$

Optimal arrangement of jobs: $X := \text{Maximize}(D, x)$

$X^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	0	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	1	0	0	1	0	0	0

Values of optimal plan for arrangement of jobs, as 5x5 matrix:

$$Xr := \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 \\ X_6 & X_7 & X_8 & X_9 & X_{10} \\ X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{16} & X_{17} & X_{18} & X_{19} & X_{20} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} \end{pmatrix} \qquad Xr = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The greatest value of criterion function: $S := |C \cdot X|$ $\quad S = 27$

Solution verification: $(A \cdot X)^T = (1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)$

Conclusion On the basis of optimal solution, it has been stated that the first working place is necessary to be given to the fifth (fictitious worker, the second to the second worker, the first post to the third worker, the fourth to the fourth one and the third should be given to the fifth worker. The first worker will not be employed.

3.4 The open problem of jobs disposition [AP4a_2.mcd]

Example In an engine room of a working organization, it is necessary to perform four tasks. These tasks can be done on six machines. Every machine can work every task, but only one at the some time. With these tasks it is necessary to work four machines, and two to separate for other tasks. The time for processing works on machines/hour/ is given in the table T.3.5.

T.3.5

Work places / *Machines*	T1	T2	T3	T4
M1	9	12	7	12
M2	14	10	9	11
M3	8	15	11	15
M4	12	13	8	14
M5	10	11	10	10
M6	11	14	12	9

How to dispose tasks on machines so the used time for performing all tasks to be as shorter as possible?

Solution

Previously defined assignment problem is open. After adding two fictitious tasks, we get the table T.3.6, and it is the condition for forming square matrix and apparently closed assignment problem.

T.3.6

Work places / *Machines*	T1	T2	T3	T4	T5	T6
M1	9	12	7	12	0	0
M2	14	10	9	11	0	0
M3	8	15	11	15	0	0
M4	12	13	8	14	0	0
M5	10	11	10	10	0	0
M6	11	14	12	9	0	0

Value vector of points C: $(ORIGIN := 1)$

$C :=$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	9	12	7	12	0	0	14	10	9	11	0	0	8	15	11	15	0	0	12	13	8	14	0	0	10

Criterion function, as function of maximal points: $n := 6$ $m := 6$

$$j := 1 .. n \cdot m \qquad D(x) := \sum_j \left(C^T\right)_j \cdot x_j$$

Complete matrix of limit (ations) coefficients is formed by opening and filling in blank table. We get Input Table through Component Wizard from the menu Insert ❍ Component.

A :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0
8	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0
9	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
10	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
11	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0
12	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1

Limit factor by which is defined that only one worker can work at one machine and that only one job can be given to one worker.

B :=

	1
1	1
2	1
3	1
4	1
5	1

One initial value (the last one): $x_{36} := 0$

Block for solving linear programming model: Given

System of linear equations: $A \cdot x = B$ $x \geq 0$

Optimal disposition of jobs: $X := \text{Minimize}(D, x)$

$X^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	0	0	1	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1

The least value of criterion function is: $S := |C \cdot X|$ $S = 34$

Values of optimal jobs disposition, arranged as the matrix 6x6:

$$\begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 \\ X_7 & X_8 & X_9 & X_{10} & X_{11} & X_{12} \\ X_{13} & X_{14} & X_{15} & X_{16} & X_{17} & X_{18} \\ X_{19} & X_{20} & X_{21} & X_{22} & X_{23} & X_{24} \\ X_{25} & X_{26} & X_{27} & X_{28} & X_{29} & X_{30} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Solution verification: $(A \cdot X)^T = (1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)$

Conclusion the first job we should process at the third machine, the second one at the second machine, the third at the first and the fourth on the sixth. The machines M4 and M5 will not be used for performing these jobs. The variables X_{23} and X_{30} are supplementary variables and they there point which machines will not be used. Minimal time possible for performing these jobs is 34 hours.

3.5 Disposition of students [AP5a_2.mcd]

Example At sitting for an examination was given eight question groups, and the exam take six students, after getting one group of questions. Students` level of knowledge for single groups of question is given with the next marks, students can get.

T.3.7

Question / *Student*	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
S1	6	8	9	5	7	10	6	5
S2	7	10	8	9	8	5	6	7
S3	6	6	5	5	9	9	10	10
S4	10	9	7	7	6	6	6	5
S5	9	8	5	6	6	7	7	8
S6	8	9	10	10	9	9	8	7

If each student should answer only one group of questions, determine the best possible one (with the highest average mark) and the possible worst (with the lowest average mark) success.

Solution

Previously defined assignment problem is opened. After adding two fictitious students, we get the table T.3.8, which is condition for forming square matrix and temporarily closed assignment problem.

T.3.8

Question / *Student*	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8
S1	6	8	9	5	7	10	6	5
S2	7	10	8	9	8	5	6	7
S3	6	6	5	5	9	9	10	10
S4	10	9	7	7	6	6	6	5
S5	9	8	5	6	6	7	7	8
S6	8	9	10	10	9	9	8	7
S7	0	0	0	0	0	0	0	0
S8	0	0	0	0	0	0	0	0

Previously defined assignment problem is closed, where is noticeable square matrix 8x8.

Value vector of the point C: $(\mathrm{ORIGIN} := 1)$

C :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	6	8	9	5	7	10	6	5	7	10	8	9	8	5	6	7	6	6	5	5	9	9	10

Criterion function as the function of maximal points: $n := 8$ $m := 8$

$$j := 1 .. n \cdot m \qquad D(x) := \sum_j \left(C^T\right)_j \cdot x_j$$

The complete matrix of limits coefficients is being formed by opening and filling in blank table. We get Input Table through the dialogue Component Wizard from the menu Insert ❍ Component.

Table A value: $(n + m) \cdot n \cdot m = 1024$ vector B value: $m + n = 16$

A :=

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Limit vector that has defined that only one worker can work at one machine and that only one job may be disposed to one worker:

B :=

	1
1	1
2	1
3	1
4	1

π **Finding the most favourable combination**

One initial value (last): $x_{64} := 0$

Block for solving model of linear programming: Given

Linear equations system: $A \cdot x = B$ $\quad x \geq 0$

Optimal distribution of jobs: $X := \text{Maximize}(D, x)$

$X^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1

Optimal plan values for jobs distribution, as the matrix 8x8:

The least value of criterion function: $S := |C \cdot X|$ $\quad S = 58$

$$Xr := \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{17} & X_{18} & X_{19} & X_{20} & X_{21} & X_{22} & X_{23} & X_{24} \\ X_{25} & X_{26} & X_{27} & X_{28} & X_{29} & X_{30} & X_{31} & X_{32} \\ X_{33} & X_{34} & X_{35} & X_{36} & X_{37} & X_{38} & X_{39} & X_{40} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} & X_{47} & X_{48} \\ X_{49} & X_{50} & X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{57} & X_{58} & X_{59} & X_{60} & X_{61} & X_{62} & X_{63} & X_{64} \end{pmatrix} \qquad Xr = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution verification: $(A \cdot X)^T = (1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1)$

In view of optimal solution the next results are concluded concerning maximum, when there are being combined group of questions and the next structure students:

T.3.9

Student	S1	S2	S3	S4	S5	S6
Group of question	Q6	Q2	Q7	Q1	Q8	Q4
Mark	10	10	10	10	8	10

The highest average mark is: $\dfrac{S}{m-2} = 9.667$

One initial value (last): $x_{64} := 0$

Block for solving linear programming models: Given

Linear equations system: $A \cdot x = B \qquad x \geq 0$

Optimal disposition of jobs: $X := \text{Minimize}(D, x)$

$X^T =$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
1	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0

The values of optimal plan for jobs disposition, as the matrix 8x8:

The least value of criterion function: $S := |C \cdot X| \qquad S = 34$

$$Xr := \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & X_8 \\ X_9 & X_{10} & X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} \\ X_{17} & X_{18} & X_{19} & X_{20} & X_{21} & X_{22} & X_{23} & X_{24} \\ X_{25} & X_{26} & X_{27} & X_{28} & X_{29} & X_{30} & X_{31} & X_{32} \\ X_{33} & X_{34} & X_{35} & X_{36} & X_{37} & X_{38} & X_{39} & X_{40} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} & X_{47} & X_{48} \\ X_{49} & X_{50} & X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} \\ X_{57} & X_{58} & X_{59} & X_{60} & X_{61} & X_{62} & X_{63} & X_{64} \end{pmatrix} \qquad Xr = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution verification: $(A \cdot X)^T = (1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1)$

On the basis of optimal solution, there are stated the next results referred to minimum, when one combines group of questions and the students of the next structure:

T.3.10

Student	S1	S2	S3	S4	S5	S6
Group of question	Q8	Q6	Q4	Q7	Q3	Q1
Mark	5	5	5	6	5	8

The least average mark is: $\frac{S}{m-2} = 5.667$

4. NONLINEAR PROGRAMMING

4.1 A model of nonlinear programming [NP1.mcd]

Example Solve atask of non-linear programming on the basis of the model with three variables and charges function (criterion), through method with of finding unconditional extreme of charges function T(x1,x2,x3). Define stationary points (vectors) of optimum and define their nature.

$$T(x1, x2, x3) := 4 \cdot x1^2 + \frac{1}{12} \cdot x2^3 + 7 \cdot x3^2 - x2 - 2 \cdot x1 + \frac{7}{2} \cdot x3 + 7$$

Solution

Necessary conditions for determing stationary points are being found in view of partial derivations for variables: x1, x2 i x3.

$$\frac{\partial}{\partial x1}T(x1,x2,x3) \rightarrow 8\cdot x1 - 2$$

$$\frac{\partial}{\partial x2}T(x1,x2,x3) \rightarrow \frac{1}{4}\cdot x2^2 - 1$$

$$\frac{\partial}{\partial x3}T(x1,x2,x3) \rightarrow 14\cdot x3 + \frac{7}{2}$$

For determing stationary vectors $x^{<1>}$ and $x^{<2>}$ previously found derivations are equaled with zero:

$$ORIGIN := 1 \qquad Given$$

$$8\cdot x1 - 2 = 0 \qquad \frac{1}{4}\cdot x2^2 - 1 = 0 \qquad 14\cdot x3 + \frac{7}{2} = 0$$

$$MinErr(x1,x2,x3) \rightarrow \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 2 & -2 \\ \frac{-1}{4} & \frac{-1}{4} \end{pmatrix}$$ what can be presented by the matrix: $$x := \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ 2 & -2 \\ \frac{-1}{4} & \frac{-1}{4} \end{pmatrix}$$

In view of previous results, we find that there are two stationary function points, presented through vectors:

$$x^{\langle 1\rangle} = \begin{pmatrix} 0.25 \\ 2 \\ -0.25 \end{pmatrix} \qquad x^{\langle 2\rangle} = \begin{pmatrix} 0.25 \\ -2 \\ -0.25 \end{pmatrix}$$

$$x_{1,1} = 0.25 \qquad x_{1,2} = 0.25$$

$$x_{2,1} = 2 \qquad x_{2,2} = -2$$

$$x_{3,1} = -0.25 \qquad x_{3,2} = -0.25$$

By derivation of higher progression and mixed derivations of many variables, are being stated its characteristics in environment of stationary points. These derivations can be directly arranged in determinants for stationary points:

$$\mathrm{det1}(x1,x2,x3) := \left|\frac{\partial^2}{\partial x1^2}T(x1,x2,x3)\right| \to 8$$

$$\mathrm{det2}(x1,x2,x3) := \left|\begin{pmatrix} \frac{\partial^2}{\partial x1^2}T(x1,x2,x3) & \frac{\partial}{\partial x1}\frac{\partial}{\partial x2}T(x1,x2,x3) \\ \frac{\partial}{\partial x2}\frac{\partial}{\partial x1}T(x1,x2,x3) & \frac{\partial^2}{\partial x2^2}T(x1,x2,x3) \end{pmatrix}\right| \to 4\cdot x2$$

$$\mathrm{det3}(x1,x2,x3) := \left|\begin{pmatrix} \frac{\partial^2}{\partial x1^2}T(x1,x2,x3) & \frac{\partial}{\partial x1}\frac{\partial}{\partial x2}T(x1,x2,x3) & \frac{\partial}{\partial x1}\frac{\partial}{\partial x3}T(x1,x2,x3) \\ \frac{\partial}{\partial x2}\frac{\partial}{\partial x1}T(x1,x2,x3) & \frac{\partial^2}{\partial x2^2}T(x1,x2,x3) & \frac{\partial}{\partial x2}\frac{\partial}{\partial x3}T(x1,x2,x3) \\ \frac{\partial}{\partial x3}\frac{\partial}{\partial x1}T(x1,x2,x3) & \frac{\partial}{\partial x3}\frac{\partial}{\partial x2}T(x1,x2,x3) & \frac{\partial^2}{\partial x3^2}T(x1,x2,x3) \end{pmatrix}\right| \to 56\cdot x2$$

Concrete values of determinants for stationary points, can be stated through substitution of values for numerical elements of the first vector, then the second one. In view of that follow the results:

$$\begin{pmatrix} \mathrm{det1}(x_{1,1},x_{2,1},x_{3,1}) \\ \mathrm{det2}(x_{1,1},x_{2,1},x_{3,1}) \\ \mathrm{det3}(x_{1,1},x_{2,1},x_{3,1}) \end{pmatrix} \to \begin{pmatrix} 8 \\ 8 \\ 112 \end{pmatrix} \qquad \begin{pmatrix} \mathrm{det1}(x_{1,2},x_{2,2},x_{3,2}) \\ \mathrm{det2}(x_{1,2},x_{2,2},x_{3,2}) \\ \mathrm{det3}(x_{1,2},x_{2,2},x_{3,2}) \end{pmatrix} \to \begin{pmatrix} 8 \\ -8 \\ -112 \end{pmatrix}$$

As all values are determinants (det1, det2 i det3) > 0, for $x^{<1>}$ here results that the function $T(x_{1,1},x_{2,1},x_{3,1})$ has local minimum with numerical value:

$$T(x_{1,1},x_{2,1},x_{3,1}) = 4.979$$

Through similar procedure, by substitution of the vector $x^{<2>}$, we are testing the amount of criterion changes function, in this stationary point:

$$T(x_{1,2},x_{2,2},x_{3,2}) = 7.646$$

As: $T(x_{1,2},x_{2,2},x_{3,2}) > T(x_{1,1},x_{2,1},x_{3,1}) = 1$ the solution is verified, so it comes that the first point $T(x_{1,1},x_{2,1},x_{3,1})$ is global minimum of charges function.

4.2 Profit maximization through nonlinear programming method [NP2.mcd]

Example On the basic of defined functions R(q) – incomes and T(q) – costs [2]:

a) Define optimal volume of sale for profit maximization.
b) State value of profit for minimal average costs.
c) Check and design on a chart functions of income, costs and profit.
d) How great is interval of great power?

Solution

Function of income: $R(q) := 400\cdot q - q^2$

Function of costs: $T(q) := 3\cdot q^2 + 1200$

Profit is defined as difference of total incomes and total costs:

$$\Pi(q) := R(q) - T(q) \rightarrow 400\cdot q - 4\cdot q^2 - 1200$$

The conditions achieving maximum of profit function are that the first derivation should be zero and the second one negative:

$$\frac{d}{dq}\Pi(q) \rightarrow 400 - 8\cdot q \qquad \frac{d^2}{dq^2}\Pi(q) \rightarrow -8$$

Consequently, profit maximum is achieved when the volume of production (sale) is 50, so then is realized the profit of 8800 /nj/.

$$\frac{d}{dq}\Pi(q) = 0 \text{ solve}, q \rightarrow 50 \qquad \Pi(50) = 8800$$

Minimal average costs can present criterion for production optimization. In the concrete case they are achieved through the next procedures:

$$t(q) := \frac{T(q)}{q} \qquad t(q) \rightarrow \frac{\left(3\cdot q^2 + 1200\right)}{q} \qquad \frac{d^2}{dq^2}t(q) \text{ simplify } \rightarrow \frac{2400}{q^3}$$

$$\frac{d}{dq}t(q) = 0 \text{ solve}, q \rightarrow \begin{pmatrix} 20 \\ -20 \end{pmatrix} \qquad \Pi(20) = 5200$$

For volume of sale (production) of q=20 product units, we can realize minimal average costs, and the profit then realized is 5200 /nj/.

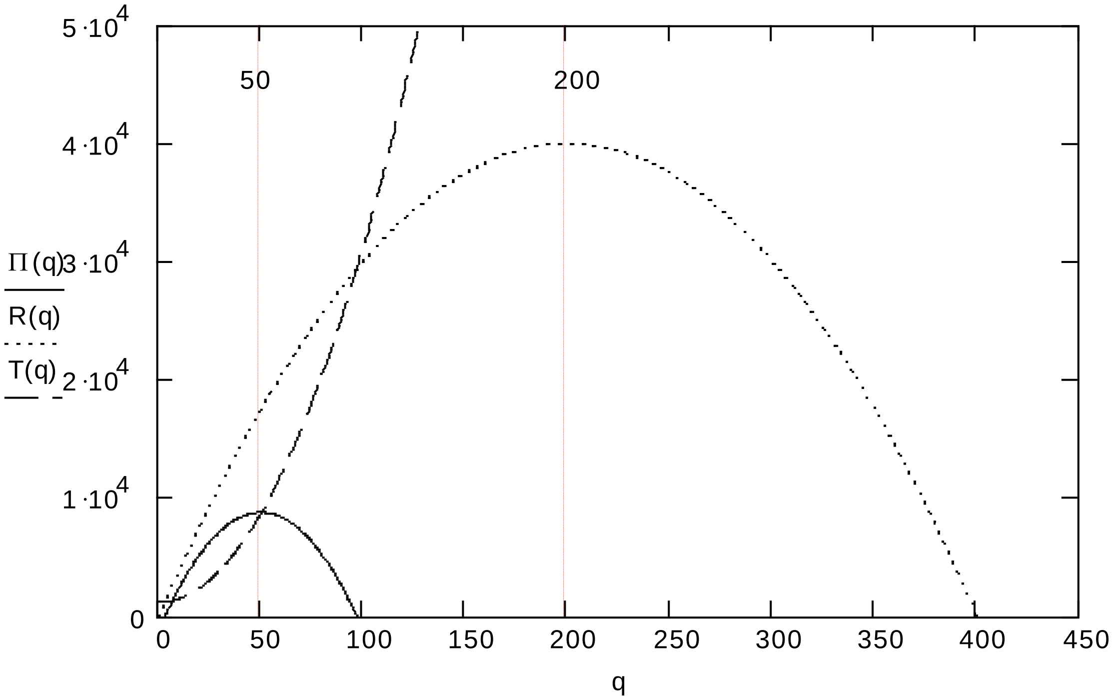


Fig. 4.1 *The chart of functions for profit, costs and income*

π **Points of section for the functions R(q) and T(q)**

Initial value of solution: $q := 50$

Solution of functions cross-section: $Q1 := \mathrm{root}(R(q) - T(q), q)$ $Q1 = 96.9$

Verification of functions values: $\begin{pmatrix} R(Q1) \\ T(Q1) \end{pmatrix} = \begin{pmatrix} 29371.247 \\ 29371.247 \end{pmatrix}$

One should note that the first coordinates of these points are exactly also the first coordinates of zero value of profit function.

Points of selection for the functions R(q) and P(q)

Solution of functions cross-selection: $Q2 := \mathrm{root}(R(q) - \Pi(q), q)$ $Q2 = -20i$

Verification of functions values: $\begin{pmatrix} R(Q2) \\ \Pi(Q2) \end{pmatrix} = \begin{pmatrix} 400 - 8000i \\ 400 - 8000i \end{pmatrix}$

In relation of the two functions R(q) and P(q) we can not find real solution of cross-selection, but only complex (imaginary). Consequently, it follows that it is impossible that profit should be equal to incomes.

π **Points of section for the functions T(q) and P(q)**

Solution of functions cross-selection: $Q3 := \mathrm{root}(T(q) - \Pi(q), q)$ $Q3 = 50.33$

Verification of functions value: $\begin{pmatrix} \Pi(Q3) \\ T(Q3) \end{pmatrix} = \begin{pmatrix} 8799.562 \\ 8799.562 \end{pmatrix}$

Interval of earning power is the interval for which profit is positive, or alternative field where incomes are greater than costs:

$\Pi(Q) := 400 \cdot Q - 4 \cdot Q^2 - 1200$ $A := \Pi(Q) \text{ coeffs}, Q \rightarrow \begin{pmatrix} -1200 \\ 400 \\ -4 \end{pmatrix}$

$q := \mathrm{polyroots}(A)$ $q = \begin{pmatrix} 3.096 \\ 96.904 \end{pmatrix}$ $q_0 = 3.096$ $q_1 = 96.904$

In concrete example it is the interval $[q_0,q_1]$. In that interval company realizes its gaining business (operations). Consequently, it is exactly the field between two zeros of profit function, respectively the field between two points of section of total income function and total costs.

4.3 Conditions of production optimality (optimum) [NP3.mcd]

Example For the product *x* are given demand function *x*(p) and costs function C(*x*). Determine production for which is being realized maximal profit and its amount.

Starting data: $a := -\frac{1}{4}$ $b := 1200$ $c := 2$ $d := 900$

Demand function: $x(p) := a \cdot p + b$

Costs function: $C(x) := c \cdot x^2 + d$

Inverse form of demand function: $p(x) := a \cdot p + b - x = 0 \text{ solve}, p \rightarrow 4800 - 4 \cdot x$

Or by redefinition (only change of marks, $x \rightarrow X$): $p(X) \rightarrow 4800 - 4 \cdot X$

In come function is then: $P(X) := X \cdot p(X) \rightarrow X \cdot (4800 - 4 \cdot X)$

Redefinition of costs function: $C(X) \rightarrow 2 \cdot X^2 + 900$

Then profit function is defined as:

$$D(x) := P(x) - C(x) \text{ simplify } \rightarrow 4800 \cdot x - 6 \cdot x^2 - 900$$

Izvodna funkcija dobiti je: $D'(x) := \frac{d}{dx} D(x) \rightarrow 4800 - 12 \cdot x$

Solving derived equation, we get a solution *x*o as optimal value of production:

$$\text{Given} \qquad D'(x) = 0 \qquad xo := \text{Find}(x) \rightarrow 400$$

So, maximal profits: $D(xo) \rightarrow 959100$

The interval of earning power in determined from the conditions: $\text{Given} \qquad D(x) = 0$

Solving square equation we get solution vector:

$$x := \text{Find}(x) \rightarrow \left(400 + 5 \cdot \sqrt{6394} \quad 400 - 5 \cdot \sqrt{6394}\right) \qquad x^T = \begin{pmatrix} 799.812 \\ 0.188 \end{pmatrix} \qquad xr := x^T$$

The earning power is between the values:

$$xr_0 = 799.812 \quad \text{i} \quad xr_1 = 0.188$$

4.4 Utility function-optimal volume of purchase of goods [NP4.mcd]

Example For the given utility function U and budget equation D with fixed budget Do determine optimal purchase volume (acquisition) of observed goods, so to realize maximal utility [23].

Starting parameters: $p1 := 45 \qquad p2 := 24$

Values of fixed budget Do: $Do := 680$

Utility function: $U(x, y) := 85 \cdot x \cdot y^2$

Lagrange's equation: $G(x, y, \lambda) := U(x, y) + \lambda \cdot (Do - D(x, y))$

Partial deviations for *x*, *y* and λ:

$$G1(x,y,\lambda) := \frac{\partial}{\partial x} G(x,y,\lambda) \qquad G1(x,y,\lambda) \rightarrow 85 \cdot y^2 - 45 \cdot \lambda$$

$$G2(x,y,\lambda) := \frac{\partial}{\partial y} G(x,y,\lambda) \qquad G2(x,y,\lambda) \rightarrow 170 \cdot x \cdot y - 24 \cdot \lambda$$

$$G3(x,y,\lambda) := \frac{\partial}{\partial \lambda} G(x,y,\lambda) \qquad G3(x,y,\lambda) \rightarrow 680 - 45 \cdot x - 24 \cdot y$$

Other partial derivations are:

$$F11(x,y,\lambda) := \frac{\partial^2}{\partial x^2} G(x,y,\lambda) \qquad F11(x,y,\lambda) \rightarrow 0$$

$$F12(x,y,\lambda) := \frac{\partial}{\partial x}\frac{\partial}{\partial y} G(x,y,\lambda) \qquad F12(x,y,\lambda) \rightarrow 170 \cdot y$$

$$F21(x,y,\lambda) := \frac{\partial}{\partial y}\frac{\partial}{\partial x} G(x,y,\lambda) \qquad F21(x,y,\lambda) \rightarrow 170 \cdot y$$

$$F22(x,y,\lambda) := \frac{\partial^2}{\partial y^2} G(x,y,\lambda) \qquad F22(x,y,\lambda) \rightarrow 170 \cdot x$$

Determing optimal value of goods purchase. For initial values is formed block for calculation:

$$x := 2 \qquad y := 2 \qquad \lambda := 1 \qquad \text{Given}$$

$$G1(x,y,\lambda) = 0 \qquad G2(x,y,\lambda) = 0 \qquad G3(x,y,\lambda) = 0$$

Optimal values of the two variables (xo, yo) make:

$$\begin{pmatrix} xo \\ yo \\ \lambda o \end{pmatrix} := \text{Find}(x,y,\lambda) \qquad \begin{pmatrix} xo \\ yo \end{pmatrix} = \begin{pmatrix} 5.037 \\ 18.889 \end{pmatrix}$$

With this maximal value of utility function is: $U(xo,yo) = 152759.031$

Values of partial derivations for optimal values:

$$F11(xo,yo,\lambda) = 0 \qquad F12(xo,yo,\lambda) = 3211.111$$

$$F21(xo,yo,\lambda) = 3211.111 \qquad F22(xo,yo,\lambda) = 856.296$$

The condition that verifies maximal values of utility function is expressed by positive value of Hesian determinant.

$$H := \left|\begin{pmatrix} F11(xo, yo, \lambda) & F12(xo, yo, \lambda) & -p1 \\ F21(xo, yo, \lambda) & F22(xo, yo, \lambda) & -p2 \\ -p1 & -p2 & 0 \end{pmatrix}\right|$$

$H = 5202000$ $\qquad$ $H > 0 = 1$ (affirmatively)

From where is confirmed that Hesian determinant is positive.

4.5 Optimization of enzyme product [NP5.mcd]

Example Optimization of models for biological processes can be observed from many views. First, that during evolution has happened significant bio-chemical changes. Now researchers should determine different metabolic pathways and genetic models caused by them. The second is the method of artificial improvement of these systems, because of different demands as insulin production out of DNA resources, before it is isolated from animal glands and the like. Taking this in consideration, researchers have developed mathematical program for ethyl-alcohol optimization, glycerin (e) (propan-triol) and production of carbohydrates in Saccharomyces cerevisiae in conditions of broken of cell culture in ph4.5.

Starting parameters connected with chemical compounds, are given in the next table:

T.4.1

Chemical compound	*Concentration*	*Concentration*	*Concentration*
Internal glucose	$x1 := 0.034$	Phosphofructokinase	$x8 := 31.7$
Glucose-6-phosphate	$x2 := 1.01$	Glyceraldehyde-3-phosphate dehydrogenase	$x9 := 49.9$
Fructose-1,6-diphosphate	$x3 := 9.144$	Pyruvate kinase	$x10 := 3440$
Phospheoenolpyruvate	$x4 := 0.0095$	Polysaccharide production	$x11 := 14.31$
Adenozin-tri-fosfat (ATP)	$x5 := 1.1278$	Glycerol production:	$x12 := 203$
Glucose uptake	$x6 := 19.7$	ATPase	$x13 := 25.1$
Enzim hexokinase	$x7 := 68.5$	NAD/NADH ratio	$x14 := 0.042$

Solution

Exact model with limit functions:

Given

Sugar transport: $x1 = 0.8122 \cdot x2^{-0.2344} \cdot x6 - 2.8632 \cdot x1^{0.7464} \cdot x5^{0.0243} \cdot x7$

Hexokinase enzyme:

$$x2 = 2.8632 \cdot x1 \cdot x5^{0.0243} \cdot x7 - 0.5236 \cdot x2^{0.735} \cdot x5^{-0.394} \cdot x8^{0.999} \cdot x11^{0.001}$$

The kind of enzyme whose reaction causes compound breakdown (Phospofructokinase reaction).

$$x3 = 0.5232 \cdot x2^{0.7318} \cdot x5^{-0.3941} \cdot x8$$

Enzim Pyruvatekinase:

$$x4 = 0.022 \cdot x3^{0.6159} \cdot x5^{0.1308} \cdot x9 \cdot x14^{-0.6088} - 0.0945 \cdot x3^{0.05} \cdot x4^{0.533} \cdot x5^{-0.0822} \cdot x10$$

ATP (Adenozin-tri- phosphate):

$$x5 = -\left(3.2097 \cdot x1^{0.198} \cdot x2^{0.196} \cdot x5^{0.372} \cdot x7^{0.265} \cdot x8^{0.265} \cdot x11^{0.002} \cdot x13^{0.47}\right) \dots \\ + 0.0913 \cdot x3^{0.333} \cdot x4^{0.266} \cdot x5^{0.024} \cdot x9^{0.5} \cdot x10^{0.5} \cdot x14^{-0.304}$$

Finding variables for constant state of concentration:

$$K := \mathrm{Find}(x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12, x13, x14)$$

Solution of equations system:

$$K^T = (0.04 \;\; 0.8 \;\; 8.45 \;\; 0.01 \;\; 1.24 \;\; 24.47 \;\; 80.46 \;\; 20.76 \;\; 50.24 \;\; 3367.16 \;\; 14.31 \;\; 203 \;\; 24.97 \;\; 0.04)$$

The hypothesis is that optimization of ethyl-alcohol production a real function and it is necessary to invent such a value, that the flowing through would be maximal one.

$$C(x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12, x13, x14) := 0.0945 \cdot x3^{.05} \cdot x4^{.533} \cdot x5^{-.0822} \cdot x10$$

The values of chemical compounds will be here initial for (non) equations of a constant state. According to Clifford (?) we can decide how much is to each enzyme allowed to change itself (with supposition that it can be physiologically and bio-technically realized). In terms of it are set limits to metabolite concentration:

Limit functions:

Given

$0.03 \le x1 \le 0.04$	$1.1 \le x5 \le 1.15$	$40 \le x9 \le 55$	$20 \le x13 \le 30$
$0.96 \le x2 \le 1.1$	$16 \le x6 \le 21$	$3200 \le x10 \le 10000$	$x14 \le 0.05$
$8.9 \le x3 \le 9.4$	$60 \le x7 \le 72$	$x11 \le 15$	
$0.07 \le x4 \le 0.11$	$28 \le x8 \le 35$	$x12 \le 210$	

Maximization of criterion function (of flow), with optimal values of arguments.

$$Z := \text{Maximize}(C, x1, x2, x3, x4, x5, x6, x7, x8, x9, x10, x11, x12, x13, x14)$$

$$Z^T = (0.034\ \ 1.1\ \ 9.4\ \ 0.11\ \ 1.1\ \ 21\ \ 68.571\ \ 30.464\ \ 55\ \ 10000\ \ 3.578\ \ 101.5\ \ 27.853\ \ 0.021)$$

5. DYNAMIC PROGRAMMING

5.1 One-dimensional process of optimal resources distribution [DP1.mcd]

Example inside a company is necessary to distribute a homogenous resource to the amount of S=8 units [19]. The resource is distributed to three plants so that separate values are integer and not greater than 5. The realized profit in single plants in linear way depends on the quantity of the distributed resource. Profit for resource unit is: for the first plant c1=4/nj/, the second plant:c2=5/nj/, the third plant:c3=2/nj/. The resource needs to be distributed so that we get maximal profit for the company on the whole. It is necessary:

a) to form mathematical model of problem, and
b) to determine optimal plan of resources distribution applying the method of dynamic programming

Starting data: Unit profits: $c1 := 4$ $c2 := 5$ $c3 := 2$

a) Mathematical model of problem makes function of aim: $F(x) := c1 \cdot x_1 + c2 \cdot x_2 + c3 \cdot x_3$

The conditions concerning limit (action) are:

$$x_1 + x_2 + x_3 \le S \qquad 0 \le x_1 \le 5 \qquad 0 \le x_2 \le 5 \qquad 0 \le x_3 \le 5$$

Marks of resource quantity:

$x3_0 := 0$ $x3_1 := 1$ $x3_2 := 2$ $x3_3 := 3$ $x3_4 := 4$ $x3_5 := 5$

Recurrent relation of direct profit from the distribution for the first resource:

$S := 0..8$ $f1_S := \mathrm{if}\left(c1\cdot S > c1\cdot x3_5, c1\cdot x3_5, c1\cdot S\right)$ $f1_S =$ $\max(f1) = 20$

$f1_S$
0
4
8
12
16
20
20
20
20

Recurrent relation of direct profit from the distribution for the second resource:

$S := 0$ $x2 := 0..S$ $f20_{x2} := c2\cdot x2 + f1_{S-x2}$ $f20_{x2} =$ $\max(f20) = 0$

$f20_{x2}$
0

$S := 1$ $x2 := 0..S$ $f21_{x2} := c2\cdot x2 + f1_{S-x2}$ $f21_{x2} =$ $\max(f21) = 5$

$f21_{x2}$
4
5

$S := 2$ $x2 := 0..S$ $f22_{x2} := c2\cdot x2 + f1_{S-x2}$ $f22_{x2} =$ $\max(f22) = 10$

$f22_{x2}$
8
9
10

$S := 3$ $x2 := 0..S$ $f23_{x2} := c2\cdot x2 + f1_{S-x2}$ $f23_{x2} =$ $\max(f23) = 15$

$f23_{x2}$
12
13
14
15

$S := 4$ $\quad x2 := 0..S$ $\quad f24_{x2} := c2 \cdot x2 + f1_{S-x2}$ $\quad f24_{x2} =$ $\quad \max(f24) = 20$

$f24_{x2}$
16
17
18
19
20

$S := 5$ $\quad x2 := 0..S$ $\quad f25_{x2} := c2 \cdot x2 + f1_{S-x2}$ $\quad f25_{x2} =$ $\quad \max(f25) = 25$

$f25_{x2}$
20
21
22
23
24
25

$S := 6$ $\quad x2 := 0..S-1$ $\quad f26_{x2} := c2 \cdot x2 + f1_{S-x2}$ $\quad f26_{x2} =$ $\quad \max(f26) = 29$

$f26_{x2}$
20
25
26
27
28
29

$S := 7$ $\quad x2 := 0..S-2$ $\quad f27_{x2} := c2 \cdot x2 + f1_{S-x2}$ $\quad f27_{x2} =$ $\quad \max(f27) = 33$

$f27_{x2}$
20
25
30
31
32
33

$S := 8$ $\quad x2 := 0..S-3$ $\quad f28_{x2} := c2 \cdot x2 + f1_{S-x2}$ $\quad f28_{x2} =$ $\quad \max(f28) = 37$

$f28_{x2}$
20
25
30
35
36
37

Vector relation of the distribution for the third resource:

$f30 := c3 \cdot x3_0 + \max(f20)$ $\quad f30 = 0 \quad \max(f30) = 0$

$$f31 := \begin{pmatrix} c3 \cdot x3_0 + \max(f21) \\ c3 \cdot x3_1 + \max(f20) \end{pmatrix} \to \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$\max(f31) = 5$

$$f32 := \begin{pmatrix} c3 \cdot x3_0 + \max(f22) \\ c3 \cdot x3_1 + \max(f21) \\ c3 \cdot x3_2 + \max(f20) \end{pmatrix} \to \begin{pmatrix} 10 \\ 7 \\ 4 \end{pmatrix}$$

$\max(f32) = 10$

$$f33 := \begin{pmatrix} c3 \cdot x3_0 + \max(f23) \\ c3 \cdot x3_1 + \max(f22) \\ c3 \cdot x3_2 + \max(f21) \\ c3 \cdot x3_3 + \max(f20) \end{pmatrix} \to \begin{pmatrix} 15 \\ 12 \\ 9 \\ 6 \end{pmatrix}$$

$\max(f33) = 15$

$$f34 := \begin{pmatrix} c3 \cdot x3_0 + \max(f24) \\ c3 \cdot x3_1 + \max(f23) \\ c3 \cdot x3_2 + \max(f22) \\ c3 \cdot x3_3 + \max(f21) \\ c3 \cdot x3_4 + \max(f20) \end{pmatrix} \to \begin{pmatrix} 20 \\ 17 \\ 14 \\ 11 \\ 8 \end{pmatrix}$$

$\max(f34) = 20$

$$f35 := \begin{pmatrix} c3 \cdot x3_0 + \max(f25) \\ c3 \cdot x3_1 + \max(f24) \\ c3 \cdot x3_2 + \max(f23) \\ c3 \cdot x3_3 + \max(f22) \\ c3 \cdot x3_4 + \max(f21) \\ c3 \cdot x3_5 + \max(f20) \end{pmatrix} \to \begin{pmatrix} 25 \\ 22 \\ 19 \\ 16 \\ 13 \\ 10 \end{pmatrix}$$

$\max(f35) = 25$

$$f36 := \begin{pmatrix} c3 \cdot x3_0 + \max(f26) \\ c3 \cdot x3_1 + \max(f25) \\ c3 \cdot x3_2 + \max(f24) \\ c3 \cdot x3_3 + \max(f23) \\ c3 \cdot x3_4 + \max(f22) \\ c3 \cdot x3_5 + \max(f21) \end{pmatrix} \to \begin{pmatrix} 29 \\ 27 \\ 24 \\ 21 \\ 18 \\ 15 \end{pmatrix}$$

$\max(f36) = 29$

$$f37 := \begin{pmatrix} c3 \cdot x3_0 + \max(f27) \\ c3 \cdot x3_1 + \max(f26) \\ c3 \cdot x3_2 + \max(f25) \\ c3 \cdot x3_3 + \max(f24) \\ c3 \cdot x3_4 + \max(f23) \\ c3 \cdot x3_5 + \max(f22) \end{pmatrix} \to \begin{pmatrix} 33 \\ 31 \\ 29 \\ 26 \\ 23 \\ 20 \end{pmatrix}$$

$\max(f37) = 33$

$$f38 := \begin{pmatrix} c3 \cdot x3_0 + \max(f28) \\ c3 \cdot x3_1 + \max(f27) \\ c3 \cdot x3_2 + \max(f26) \\ c3 \cdot x3_3 + \max(f25) \\ c3 \cdot x3_4 + \max(f24) \\ c3 \cdot x3_5 + \max(f23) \end{pmatrix} \to \begin{pmatrix} 37 \\ 35 \\ 33 \\ 31 \\ 28 \\ 25 \end{pmatrix}$$

$\max(f38) = 37$

Direct profits of distribution in function of available resource and number of lines: $S := 0..8$

$$f1_S = 20 \qquad f1 = \begin{pmatrix} 0 \\ 4 \\ 8 \\ 12 \\ 16 \\ 20 \\ 20 \\ 20 \\ 20 \end{pmatrix} \qquad f2_S := \begin{array}{|c|} \hline \max(f20) \\ \hline \max(f21) \\ \hline \max(f22) \\ \hline \max(f23) \\ \hline \max(f24) \\ \hline \max(f25) \\ \hline \max(f26) \\ \hline \max(f27) \\ \hline \max(f28) \\ \hline \end{array} \qquad f2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad f3_S := \begin{array}{|c|} \hline \max(f30) \\ \hline \max(f31) \\ \hline \max(f32) \\ \hline \max(f33) \\ \hline \max(f34) \\ \hline \max(f35) \\ \hline \max(f36) \\ \hline \max(f37) \\ \hline \max(f38) \\ \hline \end{array} \qquad f3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The greatest profit is /nj/: $F(x) := 37$

Optimal resource quantities: $x_1 := 3 \qquad x_2 := 5 \qquad x_3 := 0$

The solution may be expressed as well as: $\mathrm{ORIGIN} := 2 \qquad x = \begin{pmatrix} 0 \\ 3 \\ 5 \\ 0 \end{pmatrix} \qquad F(x) = 37$

6. MULTICRITERIA OPTIMIZATION

6.1 The method of simple additive weights [MO1.mcd]

Example Choose the most acceptable variant through the method of simple additive weights provided that all [8]:

a) Criterions are of the some significance (T1),
b) Of different significance (T2).

Solution

a) Weight coefficients of criterion are of the same significance.

Linearized matrix of decision is defined as:

$$Q := \begin{pmatrix} 0.5 & 0.5 & 1 & 0.3333 & 0.5556 \\ 0.3333 & 1 & 0.6 & 1 & 0.3333 \\ 0.6667 & 0.3333 & 0.4 & 0.6667 & 0.7778 \\ 1 & 0.2 & 0.4 & 0.6667 & 1 \end{pmatrix}$$

The vector of weight coefficients is given as: $T1 := (0.2 \quad 0.2 \quad 0.2 \quad 0.2 \quad 0.2)$

Applying maximization criterion it follows that:

$$R := \begin{bmatrix} \left| T1\cdot\left(Q^T\right)^{\langle 0\rangle} \right| \\ \left| T1\cdot\left(Q^T\right)^{\langle 1\rangle} \right| \\ \left| T1\cdot\left(Q^T\right)^{\langle 2\rangle} \right| \\ \left| T1\cdot\left(Q^T\right)^{\langle 3\rangle} \right| \end{bmatrix} \rightarrow \begin{pmatrix} .57778 \\ .65332 \\ .56890 \\ .65334 \end{pmatrix}$$

the most acceptable variant is: $\max(R) = 0.65334$. $R_3 = 0.65334$

Complete formation of variants is the next: $R_3 > R_1 > R_0 > R_2 = 1$ (affirmatively)

b) Weight coefficients of criterion are of different significance:

The vector of weight coefficients is given as: $T2 := (0.4 \quad 0.2 \quad 0.2 \quad 0.1 \quad 0.1)$

The results, in other words,
the elements of the vector R are:

$$R := \begin{bmatrix} \left| T2\cdot\left(Q^T\right)^{\langle 0\rangle} \right| \\ \left| T2\cdot\left(Q^T\right)^{\langle 1\rangle} \right| \\ \left| T2\cdot\left(Q^T\right)^{\langle 2\rangle} \right| \\ \left| T2\cdot\left(Q^T\right)^{\langle 3\rangle} \right| \end{bmatrix} \rightarrow \begin{pmatrix} .58889 \\ .58665 \\ .55779 \\ .68667 \end{pmatrix}$$

Applying maximization criterion the greatest value of the vector R suits to the fourth element.

$$\max(R) = 0.68667 \qquad R_3 = 0.68667$$

The full arrangement of variants is the next one: $R_3 > R_0 > R_1 > R_2 = 1$ (affirmatively).

7. HEURISTIC RESEARCH

7.1 Optimal pipeline diameter [HR1.mcd]

Example Fix optimal pipe diameter for a pipeline through which water is pressed back from a source to a reservoir. Criterion function includes total annual charges. While searching through analyze four calculation cases. Ignore charges for water pumping. Solve the tasks for given discrete values of the diameter D_n [18]. Measurement units suit to SI system.

Starting parameters

Number of variants in analysis and calculation: $n := 1..4$

Water elevation at the source $Z1_n$ and in the reservoir $Z2_n$: $Z1_n :=$ $Z2_n :=$

$Z1_n$	$Z2_n$
125	200
125	200
125	200
125	200

Flow of water in pumping station: $Q := 0.4$

Asbestos–cement pipeline is long: $L := 3000$

Investment charges I_n and pipe diameters D_n are: $I_n :=$ $D_n :=$

I_n	D_n
6500	0.6
7000	0.7
7500	0.8
7850	0.9

Coefficients for transformation of investment charges into annual ones: $\alpha := 0.15$

1. The coefficient of useful action of pumping station: $\eta := 65\%$
2. Frictional resistance: $\lambda := 0.02$
3. Number of hours in one year: $G := 8333$
4. Gravitational constant: $g := 9.81$
5. Parameter of horse – power costs: $a := 16668$

Solution

Pipeline costs: $Tcev_n := I_n \cdot L \cdot \alpha$ $Tcev_n =$

$Tcev_n$
2925000
3150000
3375000
3532500

According to Derki-Vijbaha, formula, linear parameter is (length): $\Delta\Pi_n := \dfrac{8 \cdot \lambda \cdot L \cdot Q^2}{g \cdot \pi^2 \cdot (D_n)^5}$

$\Delta\Pi_n =$

10.201
4.72
2.421
1.343

Horse – power of pumping station: $N_n := \frac{g \cdot Q}{\eta} \cdot (Z2_n - Z1_n + \Delta\Pi_n)$

$N_n =$

514.351
481.261
467.383
460.879

The difference between water at the resource and in the reservoir: $\Delta Z_n := Z2_n - Z1_n$

$\Delta Z_n =$

75
75
75
75

Costs of independent part from diameter: $T\Delta Z_n := \frac{g \cdot Q \cdot a}{\eta} \cdot \Delta Z_n$

$T\Delta Z_n =$

7546757.54
7546757.54
7546757.54
7546757.54

Costs of dependent part from diameter: $T\Delta\Pi_n := \frac{g \cdot Q \cdot a}{\eta} \cdot \Delta\Pi_n$

$T\Delta\Pi_n =$

1026444.21
474899.16
243580.02
135169.61

Total costs: $T_n := Tcev_n + T\Delta Z_n + T\Delta\Pi_n$

$T_n =$

11498201.7
11171656.7
11165337.6
11214427.1

Checking the values of costs function, the minimal value is $T_3 = 11165337{,}6$ and to it suits the optimal value of diameter of $D_3 = 800$ mm. Thus, it is accepted that:

$T_3 = 11165337.56$ $\qquad$ $D_3 = 0.8$

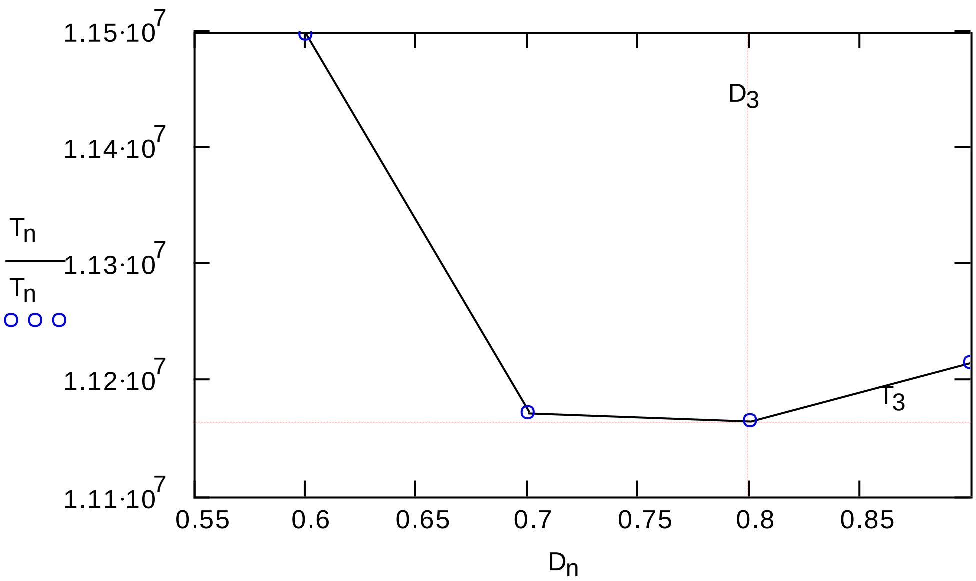


Fig. 7.1 *Dependence of total expenses of pipeline diameter*

8. MARKOV MODELS

8.1 The model for forecast consumers' orientation [MM1.mcd]

Example Orientation of clients for the products A, B and C at the *t*0 moment is presented by the vector of starting point [2].

$$V0 := (0.55 \quad 0.25 \quad 0.2) \qquad \text{or} \qquad V0 = (55 \quad 25 \quad 20)\,\%$$

Markov`s matrix of transitional probabilities is given in the form: $M := \begin{pmatrix} 0.3 & 0.5 & 0.2 \\ 0.4 & 0.1 & 0.5 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}$

With the assumption of Markov matrix stationariness, determine:

a) How many buyers decide for the products A, B and C after three year's period? Present graphically results in moments *t* = 0 year and *t* = 3 years.
b) Define balanceable (final) state of demand of these products.

Solution and verification

For previous vector V0 and matrix M follows:

$$\sum V0 = 1 \qquad \sum \left(M^T\right)^{\langle 0\rangle} = 1 \qquad \sum \left(M^T\right)^{\langle 1\rangle} = 1 \qquad \sum \left(M^T\right)^{\langle 1\rangle} = 1$$

Vectors of state:

For the first year *t*= 1: $V1 := V0 \cdot M$ $V1 = (0.305 \quad 0.42 \quad 0.275)$ $\sum V1 = 1$

For the second year *t*= 2: $V2 := V1 \cdot M^2$ $V2 = (0.303 \quad 0.389 \quad 0.308)$ $\sum V2 = 1$

For the third year *t*= 3: $V3 := V2 \cdot M^3$ $V3 = (0.307 \quad 0.379 \quad 0.314)$ $\sum V3 = 1$

Enumerating (of) state vectors: $V := \mathrm{stack}(V0, V1, V2, V3)$ $V^{\langle 0 \rangle} = \begin{pmatrix} 0.55 \\ 0.305 \\ 0.303 \\ 0.307 \end{pmatrix}$

Matrix of series now makes (progressions): $V = \begin{pmatrix} 0.55 & 0.25 & 0.2 \\ 0.305 & 0.42 & 0.275 \\ 0.303 & 0.389 & 0.308 \\ 0.307 & 0.379 & 0.314 \end{pmatrix}$

Preparation for graphical interpretation: $t := 0..3$

$A_t := \begin{array}{|c|} \hline \left(V^{\langle 0 \rangle}\right)_0 \\ \hline \left(V^{\langle 0 \rangle}\right)_1 \\ \hline \left(V^{\langle 0 \rangle}\right)_2 \\ \hline \left(V^{\langle 0 \rangle}\right)_3 \\ \hline \end{array}$ $B_t := \begin{array}{|c|} \hline \left(V^{\langle 1 \rangle}\right)_0 \\ \hline \left(V^{\langle 1 \rangle}\right)_1 \\ \hline \left(V^{\langle 1 \rangle}\right)_2 \\ \hline \left(V^{\langle 1 \rangle}\right)_3 \\ \hline \end{array}$ $C_t := \begin{array}{|c|} \hline \left(V^{\langle 2 \rangle}\right)_0 \\ \hline \left(V^{\langle 2 \rangle}\right)_1 \\ \hline \left(V^{\langle 2 \rangle}\right)_2 \\ \hline \left(V^{\langle 2 \rangle}\right)_3 \\ \hline \end{array}$

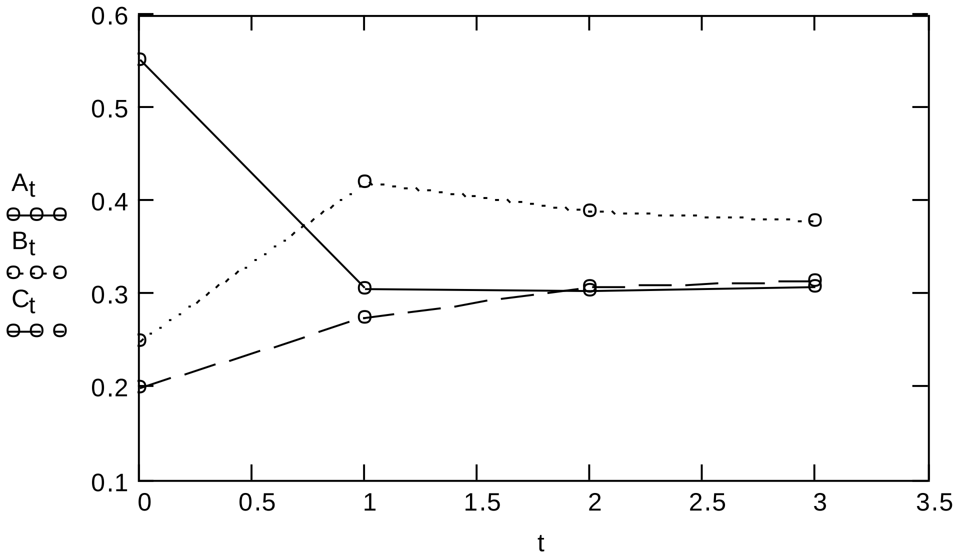


Fig. 8.1 *Graph (plot) of Markov`s probabilities for different time moments*

Conclusion It can be concluded that the product A that was in the greatest demand in the beginning of the period after the expiry of three periods falls on the third place and it points out that in that product one should not invest further. It is also concluded that one ought to invest into the product B, because it becomes leading product at the market.

π **Stable balanceable state**

Stable balanceable state is being defined on the basis of the system of matrix equations for balanceable state:

$$S \cdot M = S \qquad \sum S = 1$$

π **The solution of equations system**

Initial values: $S := \begin{pmatrix} \frac{5}{10} & \frac{4}{10} & \frac{3}{10} \end{pmatrix}$

$$\text{Given} \qquad S \cdot M - S = 0 \qquad \sum S = 1$$

$$(s1 \;\; s2 \;\; s3) := \text{MinErr}(S)$$

Solving this system of equations one gets vector of stable balanceable state W:

$$W := (s1 \;\; s2 \;\; s3) \qquad W = (0.30657 \;\; 0.37956 \;\; 0.31387)$$

With verification: $\sum W = 100\%$

Limit (bordering) matrix has come as enumerating of vectors: $G := \text{stack}(W, W, W)$

$$G = \begin{pmatrix} 0.30657 & 0.37956 & 0.31387 \\ 0.30657 & 0.37956 & 0.31387 \\ 0.30657 & 0.37956 & 0.31387 \end{pmatrix}$$

8.2 Negotiability of assets (Credits, demands) [MM2.mcd]

Example In order to apply the procedure of analysis and foreseeing of negotiability of assets, we shall suppose in this example that Markov`s matrix elements present modifications of the state of demands during one month period, as well that this is a stationary matrix. We estimate that: probability that demands with due date to one month will be cased is 0,5; probability that withholding of these demands will be written off (non-negotiable) makes 0; probability of withholding of these claims in state of demanding up to 30 days makes 0,2; while probability that these demands will pass into state of negotiability in the term of 30-90 days makes 0,3. Similarly, for demands (claims) that are in the fourth state, in other words with due date of 30-90 days, are supposed the next values of transitional probabilities [1]:

$P_{0,3} := 0.45 \qquad P_{1,3} := 0.1 \qquad P_{2,3} := 0.15 \qquad P_{3,3} := 0.3$

In view of these information's, Markov`s matrix of transitional probabilities can be presented in the form of:

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.2 & 0.3 \\ P_{0,3} & P_{1,3} & P_{2,3} & P_{3,3} \end{pmatrix}$$

In order to determine fundamental matrix we shall divide the matrix P in the way described in the chapter about vectors and matrices.

$I := \mathrm{submatrix}(P, 0, 1, 0, 1) \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad O := \mathrm{submatrix}(P, 0, 1, 2, 3) \qquad O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$A := \mathrm{submatrix}(P, 2, 3, 0, 1) \qquad A = \begin{pmatrix} 0.5 & 0 \\ 0.45 & 0.1 \end{pmatrix}$$

$$B := \mathrm{submatrix}(P, 2, 3, 2, 3) \qquad B = \begin{pmatrix} 0.2 & 0.3 \\ 0.15 & 0.3 \end{pmatrix}$$

Fundamental matrix will be: $\quad F := (I - B)^{-1} \qquad F = \begin{pmatrix} 1.359 & 0.583 \\ 0.291 & 1.553 \end{pmatrix}$

Markov`s matrix of transitional probabilities is now: $\quad K := F \cdot A \qquad K = \begin{pmatrix} 0.942 & 0.058 \\ 0.845 & 0.155 \end{pmatrix}$

The elements of the first sort of the matrix X indicate probability of demands negotiability with due date up to 30 days. So, it can be seen that probability that these claims will be cashed makes 94,2 %. Probability that these claims will be written off is 5,8%. The second sort elements that show final values of demands negotiability with due date up to 30-90 days, have similar interpretation. It is expected that 84,5% of these demands will be cashed, while 15,5% will remain uncollected. If we suppose that company at certain moment has 10000 dinars of claims with due date up to 30 days and 40000 dinars with due date up to 30-90 days, then the vector of demands is:

$$Q := \begin{pmatrix} 10000 & 40000 \end{pmatrix}$$

Amount of demands that will be cashed (written off) at the end of one month, we get on the basis:

$$Qc := Q \cdot K \qquad Qc = \begin{pmatrix} 43203.9 & 6796.1 \end{pmatrix}$$

As it can be seen, from the total amount of 50000 dinars, that are company claims from this example, we may expect that 43204 will be cashed, while 6796 dinars of demands will be written off. The same result we can get in view of the vector equation:

$$Qc := Q\cdot\left[\left[(\mathrm{submatrix}(P,0,1,0,1) - \mathrm{submatrix}(P,2,3,2,3))^{-1}\right]\cdot \mathrm{submatrix}(P,2,3,0,1)\right]$$

$$Qc = (\,43203.9 \quad 6796.1\,)$$

9. MASS QUEUEING

9.1 The system of queueing M/G/1/ infinite [MQ1.mcd]

Example In a stockroom for heavy machines a high crane is efficient 75% of time. Studying the time for hoist and carrying has given the mean time of 10,5 minutes and standard aberration 8,8 minutes [26].

a) How frequent is average series of calling for service (hoist, carrying and lowering by crane) and what time on the average is necessary for service.
b) If average service time would be 8 minutes with standard aberration of 6 minutes, how are in that case reduced wanted characteristics?

Solution

It is theoretically M/G/1 endless process. The calculation parameters are (the) next:

$$\rho 1 := 0.75 \qquad \tau 1 := 10.5\cdot min \qquad \tau 2 := 8\cdot min \qquad \sigma 1 := 8.8\cdot min \qquad \sigma 2 := 6\cdot min$$

Service frequency: $\mu 1 := \dfrac{1}{\tau 1}$ $\qquad \mu 1 = 5.71\,hr^{-1}$

Flow intensity: $\lambda := \rho 1\cdot\mu 1$ $\qquad \lambda = 4.29\,hr^{-1}$

Mean time of crane delay (retention): $t1 := \dfrac{\rho 1\cdot\left(1+\mu 1^{2}\cdot\sigma 1^{2}\right)}{2\cdot\mu 1\cdot(1-\rho 1)}$ $\qquad t1 = 26.81\,mir$

If delay time is 8 minutes, then is:

$$\mu 2 := \frac{1}{\tau 2} \qquad \mu 2 = 7.5\,hr^{-1} \qquad \rho 2 := \frac{\lambda}{\mu 2} \qquad \rho 2 = 0.571$$

so crane utilization is reduced on 57,1% of time, and makes:

$$t2 := \frac{\rho 2 \cdot \left(1 + \mu 2^2 \cdot \sigma 2^2\right)}{2 \cdot \mu 2 \cdot (1 - \rho 2)} \qquad t2 = 8.33\,\text{mir}$$

then is: $\Delta t := t1 - t2$ $\qquad \Delta t = 18.48\,\text{mir}$

and that means that waiting time is reduced for 18,5 minutes (70%).

9.2 Sixchannel model of mass queueing [MQ2.mcd]

Example Sixchannel system of mass queueing of buyers functions on the assumption that average speed of queueing on/ per channel m= 45 clients per hour, and that average speed of coming 1=225 clients per hour (on all k= 6 channels). Queueing charges on one channel make C1 = 21 /mu/hour/. Determine basic parameters of system working regime and check validity of opeming the seventh service channel in case that the cost made by clients waiting in front of the channel, C2 = 12 /mu/hour/. Here are known the next data:

$$\mu := 45 \cdot hr^{-1} \qquad \lambda := 225 \cdot hr^{-1} \qquad c1 := 21 \qquad c2 := 12 \cdot hr^{-1} \qquad k := 6$$

Solution

Queueing factor on (per) channel $\rho := \frac{\lambda}{\mu} \quad \rho = 5$

Queueing factor of the system $\rho s := \frac{\rho}{k} \qquad \rho s = 0.833$

Probability of the state when in the queueing system there are no clients, and in the stationary work system, is equal:

$$P_0 := \left(\sum_{x=0}^{k} \frac{\rho^x}{x!} + \frac{\rho^k}{k!} \cdot \frac{\rho s}{1 - \rho s}\right)^{-1} \qquad P_0 = 0.00451$$

For all values x < k, state probabilities in stationary work system are determined through the relation:

$$x := 0 .. k \qquad P_x := \frac{\rho^x}{x!} \cdot P_0$$

in single form:

$P_0 = 0.00451$ $\qquad P_3 = 0.09400$ $\qquad P_6 = 0.09792$

$P_1 = 0.02256$ $\qquad P_4 = 0.11750$

$P_2 = 0.05640$ $\qquad P_5 = 0.11750$

or in vector form:
$$\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{pmatrix} = \begin{pmatrix} 0.00451 \\ 0.02256 \\ 0.05640 \\ 0.09400 \\ 0.11750 \\ 0.11750 \\ 0.09792 \end{pmatrix}$$

Average number of clients that waits for service: $G := \frac{\rho^6}{k!} \cdot \frac{\rho s}{(1-\rho s)^2} \cdot P_0$ $\quad G = 2.94$

Average total of clients that waits is queued in service channels:

$$Q := 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + 4 \cdot P_4 + 5 \cdot P_5 + 6 \cdot (1 - P_0 - P_1 - P_2 - P_3 - P_4 - P_5) \qquad Q = 5$$

Average number of clients delaying in the queueing system: $R := G + Q$ $\quad R = 7.938$

Average time of clients waiting in queue $\tau := \frac{G}{\lambda}$ $\quad \tau = 47.001\,s$ $\quad \tau = 0.013\,hr$

Average time of clients delayin in the system (waiting and queueing)

$$T := \tau + \frac{1}{\mu} \qquad T = 0.0353\,hr \qquad T = 2.12\,mir$$

Specific costs of delaying on account of clients waiting for service

$$Tv := c2 \cdot G \cdot \tau \qquad Tv = 0.46$$

Then follows the statement that it is not economically to open the seventh working place for the job of queueing clients.

10. SIMULATION MODELING

10.1 Random number generation [SM16.mcd]

In order to perform efficiently numerical modeling of random phenomena that can be described by functions, it is necessary to possess random number generator. As basis often is used random number uniformly disposed in standard interval [0,1]. In that case one may use Mathcad function

runif(N,0,1). With the help of incorporated function *rnd*(θ) is generated random number, uniformly arranged in interval [0,θ], where limit value ɵ is real number, greater than zero. Such method of generation is a more general case than it is offered by generation function *runif*. Application of this function is especially found in process modeling in technique (technology), where there are vectors of disturbing action on the system of random character. Meanwhile, by its nature the numbers of the type *rnd*(θ) are pseudorandom, but as statistical tests indicate, they are verified as adequate replacement for real random numbers.

Example Random numbers sometimes ought to be disposed in an interval different than [0,1], or there are necessary only integer numbers. By generation of twelve uniformly arranged random numbers in the interval [a,b] are achieved by the next model.

$$a := -3 \qquad b := 3 \qquad n := 0..11 \qquad S_n := a + (b-a)\cdot rnd(1)$$

$$S^T = (-2.99 \;\; -1.84 \;\; 0.51 \;\; -0.9 \;\; 1.94 \;\; -1.96 \;\; 1.26 \;\; -1.18 \;\; -2.45 \;\; -2.12 \;\; 2.93 \;\; -2.29)$$

if integer numbers are wanted, then is applied the function: $S_n := \mathrm{round}[a + (b-a)\cdot rnd(1)]$

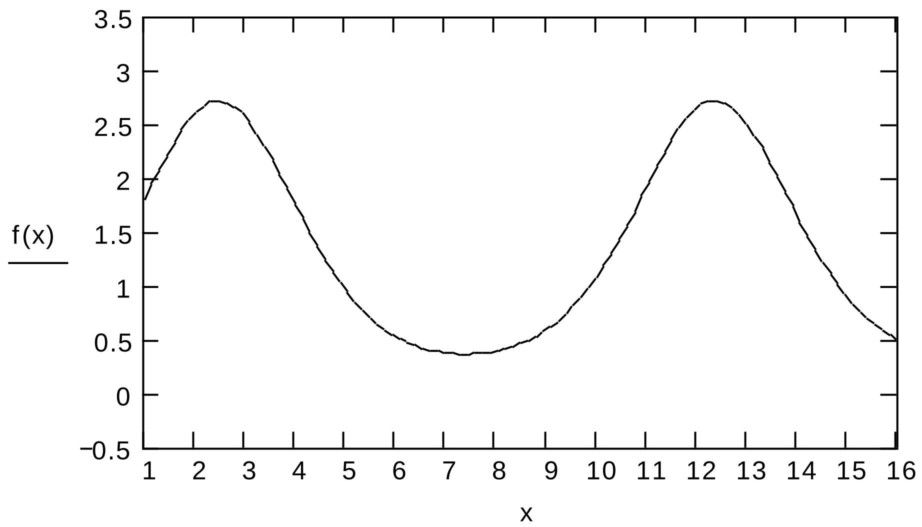
$$S^T = (-3 \;\; -1 \;\; 2 \;\; 2 \;\; 1 \;\; 1 \;\; 3 \;\; -1 \;\; 2 \;\; 0 \;\; 2 \;\; -3)$$

π **The model with deterministic function f(x)**

Example The function *rnd*(θ) is often used in an operand of a complex function. How much is the trend of that complex function. How much is the trend of that complex function “disturbed” by the influence of the function *rnd*(θ), directly depends on the value ɵ, what may be noted in the next examples.

Incremental scaling of the axis of abscissae: $x := 1, 1.15..16$

Chosen deterministic function: $f(x) := \exp\left(\sin\left(2\cdot\frac{x}{\pi}\right)\right)$



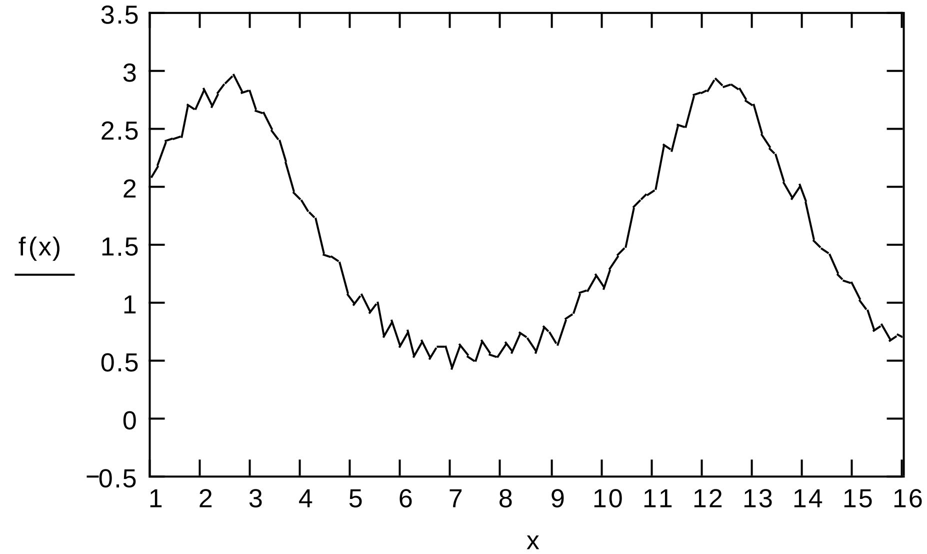
f(x) =

1.812
1.951
2.088
2.22
2.343
2.453
2.548
2.625
2.679
2.711
2.718
2.7

Fig. 10.1 *Graph (plot)of the function with determinated trend*

π Model of the function f(x) with weaker trend disturbance

The function f(x) with added operand *rnd*(θ): $f(x) := \exp\left(\sin\left(2\cdot\frac{x}{\pi}\right)\right) + \mathrm{rnd}(0.3)$



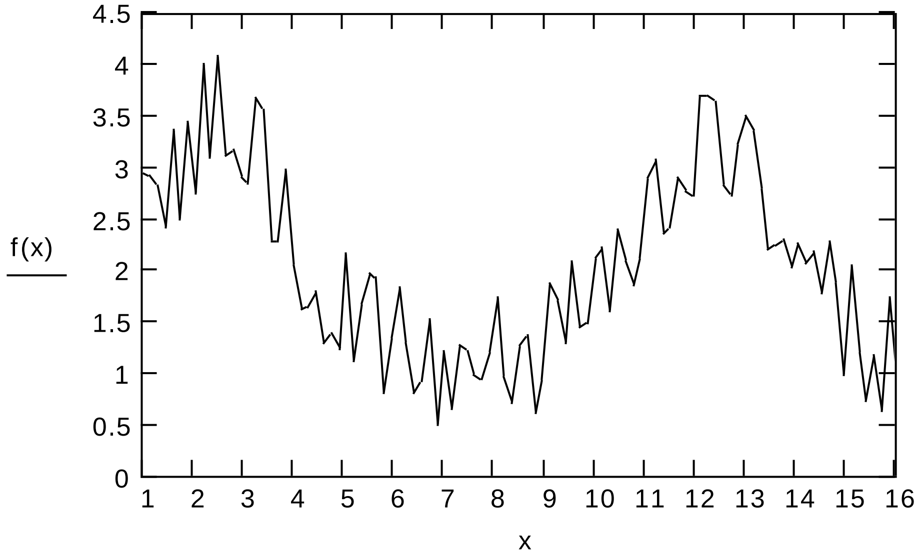
f(x) =

1.96
2.173
2.274
2.461
2.516
2.727
2.767
2.825
2.774
2.802
2.75
2.955

Fig. 10.3 *Graph (plot) of the function with random trend-variable with weaker trend*

π Model of the function f(x) with marked trend disturbance

The function f(x) with added operand *rnd*(θ): $f(x) := \exp\left(\sin\left(2\cdot\frac{x}{\pi}\right)\right) + \mathrm{rnd}(1.4)$



f(x) =

2.668
3.177
2.921
3.33
3.17
3.25
3.49
2.639
3.157
3.79
3.935
3.716

Fig. 10.4 *Graph (plot) of the function with random trend-variable with more marked disturbed trend*

Advice In order to make the procedures for random numbers more authentic, it is necessary to change periodically the value of “seed” of random numbers generator. This change is done by the option seed value for random numbers in the menu Math ❍ Options… Value “seed” is initial value of the numerical algorithm *rnd* and is defined that it is always equal or greater than one.

10.2 Simulating time reliability of technical system elements [SM2_a.mcd]

Example The elements of the technical system S1, S2 and S3 are connected series-parallel according to the fig.1. On the basis of N= 4000 simulations, calculate mean time of system work without failure and compare it with theoretical result, if for every element, times of work without failure are exponentially disposed with known intensities of failure λ□□ λ_2 and λ_3. Calculate relative aberration of theoretical and simulated mean value of time failure. Graphically present the functions of single and superponerated reliability.

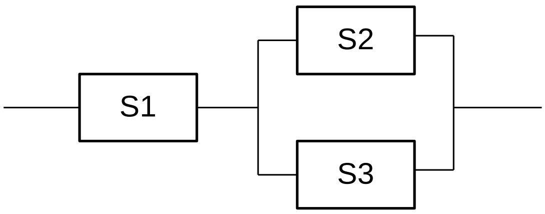


Fig. 10.5 *The chart of the system S of technical elements in parallel connection*

π **Symbolical value solution of system reliability at series-parallel connection of elements**

Functions of exponential reliability for three elements apart:

$$R1(\lambda 1, t) := e^{-\lambda 1 \cdot t} \qquad R2(\lambda 2, t) := e^{-\lambda 2 \cdot t} \qquad R3(\lambda 3, t) := e^{-\lambda 3 \cdot t}$$

Total superponerated reliability function:

$$R(\lambda 1, \lambda 2, \lambda 3, t) := e^{-\lambda 1 \cdot t} \cdot \left[1 - \left(1 - e^{-\lambda 2 \cdot t}\right) \cdot \left(1 - e^{-\lambda 3 \cdot t}\right)\right]$$

Symbolical solution for mean failure time of system at this combined connection of elements S1, S2 and S3:

$$\int_0^{\infty} R(\lambda1,\lambda2,\lambda3,t)\,dt \;\Bigg|\; \begin{array}{l}\text{assume}, \lambda1 > 0\\ \text{assume}, \lambda2 > 0\\ \text{assume}, \lambda3 > 0\end{array} \rightarrow \frac{\left(\lambda1^2 + 2\cdot\lambda1\cdot\lambda2 + 2\cdot\lambda1\cdot\lambda3 + \lambda2^2 + \lambda2\cdot\lambda3 + \lambda3^2\right)}{(\lambda1+\lambda3)\cdot(\lambda1+\lambda2)\cdot(\lambda1+\lambda2+\lambda3)}$$

- **Theoretical value of system reliability at parallel connection of elements**

Failure intensities of single elements make:

S1: $\lambda1 := 0.35\cdot10^{-3}$ S2: $\lambda2 := 0.28\cdot10^{-3}$ S3: $\lambda3 := 0.32\cdot10^{-3}$

Total theoretical reliability for parallel connection:

$$R(t) := R1(\lambda1,t)\cdot[1-(1-R2(\lambda2,t))\cdot(1-R3(\lambda3,t))]$$

Theoretical mean time of system reliability at parallel connection of elements S1, S2 and S3:

$$Trp := \int_0^{\infty} R(t)\,dt \qquad \text{or concretely:} \qquad Trp \rightarrow 2027.2073217870547014$$

Mean time of elements reliability 1: $\int_0^{\infty} e^{-\lambda1\cdot t}\,dt \rightarrow 2857.1428571428571429$

Mean time of elements reliability 2: $\int_0^{\infty} e^{-\lambda2\cdot t}\,dt \rightarrow 3571.4285714285714286$

Mean time of elements reliability 3: $\int_0^{\infty} e^{-\lambda3\cdot t}\,dt \rightarrow 3125.0000000000000000$

Mean time of elements reliability at series connection of elements S1, S2 and S3:

- **Simulation of system reliability at parallel connection of elements**

Simulation of reliability time for S1:

Number of simulations: $N := 4000$ $i := 0..N-1$

$T1 := \text{rexp}(N,\lambda1)$ $T1^T =$

	0	1	2	3	4	5
0	173.5	3024.5	908.3	74.5	2	3350

Mean time of failure of elements for S1: $\text{mean}(T1) = 2874.488$

Simulation of reliability time for S2:

$T2 := rexp(N, \lambda 2)$ $T2^T =$

	0	1	2	3	4	5
0	7364.8	2151.4	429.5	5456.6	7776	4018.9

Mean time of failure of elements for S2: $mean(T2) = 3569.163$

Simulation of reliability time for S3:

$T3 := rexp(N, \lambda 3)$ $T3^T =$

	0	1	2	3	4	5
0	2857.2	5425.8	1276.8	259.8	2962.9	259.7

Mean time of failure of elements for S3: $mean(T3) = 3113.725$

Simulation of reliability time at parallel connection:

$$T_i := if\left(T1_i < if\left(T2_i > T3_i, T2_i, T3_i\right), T1_i, if\left(T2_i > T3_i, T2_i, T3_i\right)\right)$$

$T^T =$

	0	1	2	3	4	5	6	7
0	173.54	3024.47	908.33	74.49	1.96	3349.97	2968.44	1761.88

Mean time of system failure at parallel connection: $mean(T) = 2040.245$

Relative aberration of theoretical and simulation value is relatively little and it makes

$$\Delta := \frac{mean(T) - Trp}{Trp} \qquad \text{tj.} \qquad \Delta = 0.643\%$$

π **Graphical interpretation of reliability function**

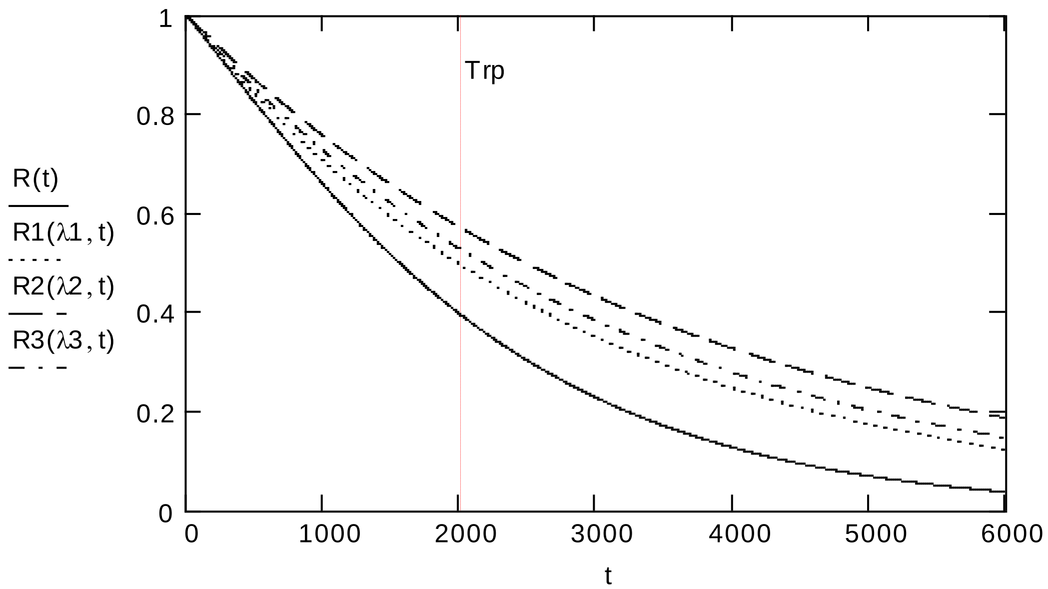


Fig. 10.6 *Reliability functions for the system with series-parallel connected elements*

As the number of replication increases, we note convergence of simulation result to very precise (one). This way of evaluation is enabled, because the results of referenced model are known, i.e. analytical one. In simpler cases of reliability modeling may be formed both models, then compare their results. In more complex cases, it is often is only possible to apply *Monte Carlo* method, so because of that reason it is called "method of the last output (solution)" [29].

11. MATRIX GAMES

11.1 Nonsingular matrix games [MG1.mcd]

Example Some two players may choose one of the next three numbers 0, 1 and 2. If addition of three numbers is even, then it goes to the second player, and if it is odd, to the first (one). Payment matrix for this game is:

$$A := \begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{pmatrix}$$

This matrix is singular, so here can not directly be applied the method of calculation for defining optimal mixed strategies. Meanwhile, if to every element of this matrix is added one, we shall get the matrix C.

$$B := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad C := A + B \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ -1 & 4 & -3 \end{pmatrix}$$

Inverse matrix of C matrix is equal:

$$C^{-1} \rightarrow \begin{pmatrix} \frac{13}{16} & \frac{-1}{8} & \frac{-7}{16} \\ \frac{-1}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{-7}{16} & \frac{3}{8} & \frac{5}{16} \end{pmatrix}$$

so the value of this modified game is:

$$V := \frac{1}{I^T \cdot C^{-1} \cdot I} \rightarrow 1$$

where is:

$$I \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

because: $I^T \cdot C^{-1} \cdot I = (1)$ in other words the value of starting game is equal to 0. optimal mixed strategies both for starting and modified game, are:

$$p := \frac{I^T \cdot C^{-1}}{I^T \cdot C^{-1} \cdot I} \rightarrow \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad q := \frac{C^{-1} \cdot I}{I^T \cdot C^{-1} \cdot I} \rightarrow \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$$

12. INVENTORY CONTROL

12.1 The calculation of optimal quantities of inventory with constant procurement [IC1.mcd]

Example A trade company plans to acquire Q= 1600 /pieces/ (of) products during period τ= 1 /year/ (365 days). The buying price of the order is K0 = 83 /mu/, while annual charges for storage k1 = 5,6 /mu/kom/. Calculate:

a) Optimal quantity qo that ought to be contained in order so that total costs of inventories would be minimized.
b) Calculate adequate quantity of products by order q1 and q2 /pieces/, if total charges of inventories increase for p= 50%. Realize comparison in relation to optimal quantity of products.
c) Give graphical presentation of flows for stockpiling and spending of product inventories for found numerical solutions.

Starting data

Annual quantity of products /pieces/:	$Q := 1600$
Plan period /years/:	$\tau := 1$
Fixed costs of inventories ordering per series /mu/:	$k0 := 83$
Costs of inventories storage /mu/pieces/year/:	$k1 := 5.6$
Percentage of increased costs for intervened storage in relation to regular one:	$p := 50 \cdot \%$

Solution

Optimal quantity of ordering /pieces/: $qo := \sqrt{\frac{2 \cdot Q \cdot k0}{\tau \cdot k1}}$ $qo = 217.78$

Optimal number of annual series /ser/years/: $n := \frac{Q}{qo}$ $n = 7.35$

Time period of one series /year/: $t := \frac{\tau}{n}$ $t = 0.136$

Minimal expenses of inventory /mu/: $F := k0 \cdot \frac{Q}{qo} + k1 \cdot \frac{qo}{2} \cdot \tau$ $F = 1219.57$

or: $minF := \sqrt{2 \cdot k0 \cdot k1 \cdot Q \cdot \tau}$ $minF = 1219.57$

Expenses increased for p.percentage /mu/: $F1 := (1 + p) \cdot F$ $F1 = 1829.36$

Equation of total expenses: $k0 \cdot \frac{Q}{q} + k1 \cdot \frac{q}{2} \cdot \tau - F1 = 0$

that is equivalent to square equation: $\tau \cdot \frac{k1}{2} \cdot q^2 - F1 \cdot q + k0 \cdot Q = 0$

Solving root of equation on the basis of equation coefficients: $q12 := \begin{pmatrix} k0 \cdot Q \\ -F1 \\ \frac{k1 \cdot \tau}{2} \end{pmatrix}$

$ORIGIN := 1$ $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} := polyroots(q12)$

Two solutions of square equation make /pieces/: $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 83.185 \\ 570.158 \end{pmatrix}$

Alternative marks: $q1 := q_1$ $q2 := q_2$

Relation of two inventory quantities: $u := \frac{q2}{q1}$ $u = 6.854$

Graphical interpretation of increased inventory costs for p= 50%.

Axis scaling of inventory quantity: $q := 0, 1 .. 650$

Function of total costs: $f(q) := k1 \cdot \frac{q}{2} \cdot \tau + k0 \cdot \frac{Q}{q}$

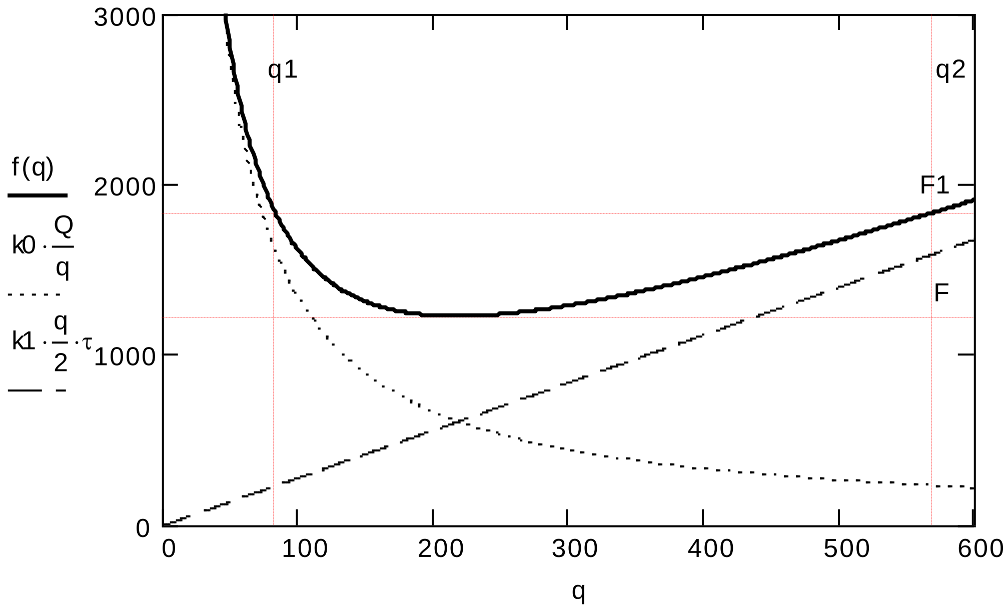


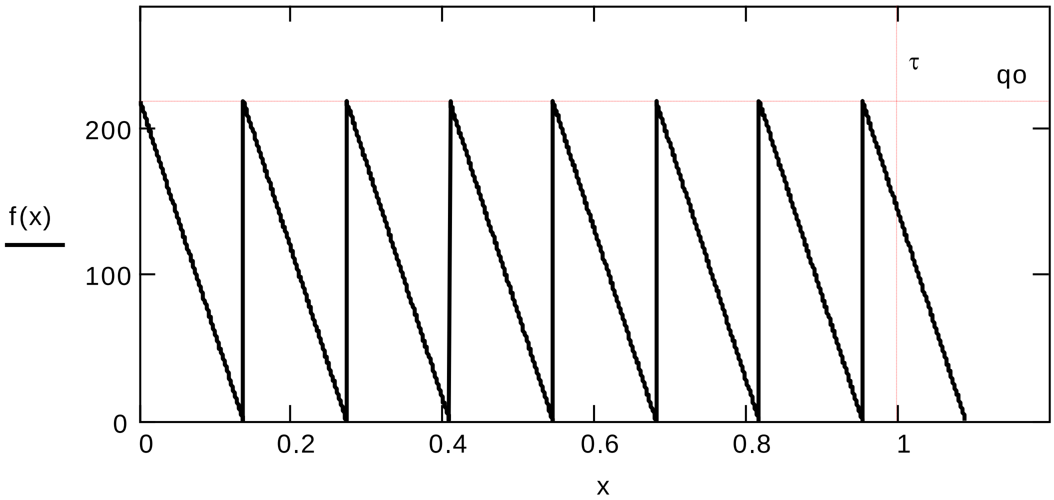
Fig. 12.1 *Functions of total and partial costs*

Scaling of independent variable: $x := 0, \frac{t}{200} .. \mathrm{ceil}(n) \cdot t$

Inventory programming in time for the first nine series:

$$f(x) := \begin{cases} qo \cdot \left(1 - \frac{x}{t}\right) & \text{if } 0 \le x \le t \\ qo \cdot \left(2 - \frac{x}{t}\right) & \text{if } t < x \le 2 \cdot t \\ qo \cdot \left(3 - \frac{x}{t}\right) & \text{if } 2 \cdot t < x \le 3 \cdot t \\ qo \cdot \left(4 - \frac{x}{t}\right) & \text{if } 3 \cdot t < x \le 4 \cdot t \\ qo \cdot \left(5 - \frac{x}{t}\right) & \text{if } 4 \cdot t < x \le 5 \cdot t \\ qo \cdot \left(6 - \frac{x}{t}\right) & \text{if } 5 \cdot t < x \le 6 \cdot t \\ qo \cdot \left(7 - \frac{x}{t}\right) & \text{if } 6 \cdot t < x \le 7 \cdot t \\ qo \cdot \left(8 - \frac{x}{t}\right) & \text{if } 7 \cdot t < x \le 8 \cdot t \end{cases}$$

In view of previous data we can plan inventory quantities for operative period, during so-called “sowlike” diagram.



Sl. 12.2 *Dijagram stanja zaliha u vremenu*

12.2 Calculation of optimal inventory quantities with intervened order [IC2.mcd]

Example In the case that demand for inventory is greater than regularly acquired quantity, calculate.

a) Optimal volume q of demand on inventories during time period t as well optimal volume of inventories p in the same time interval t. Because of that calculate intervened quantity Δ q.
b) Optimize interval cycle t(time between previous and next series), as well number of orders during time τ = 12 /months (365 days), if the next data are given:
   Q = 245000 / pieces/ - total demand during year, k1= 80 /mu/pieces/months / - unit costs of inventory maintenance, k2= 260/nj/pieces/months/ - costs that appear an account of uneven delivering of inventory, k0 = 7300 /mu/- fixed costs of series acquisition.
c) Determine necessary quantity p and Δq, if new quantity q = 3400 /pieces/, and total costs of inventory in new conditions make F=11900000 /mu/. The other parameters remain the some as under b).
d) Present graphically the case under c).

Starting data

Annual quantity of products /pieces/:	$Q := 245000$
Planned period of inventory converting /months/:	$\tau := 12$
Fixed costs of inventory per series /mu/:	$k0 := 7300$
Costs of inventory storage /mu/pieces/months/:	$k1 := 80$
Costs of storage of intervened quantity of inventory /mu/pieces/months/:	$k2 := 260$

Solution

Optimal values of inventory /pieces/: $q := \sqrt{\frac{2\cdot Q\cdot k0}{\tau\cdot k1}}\cdot\sqrt{\frac{k1+k2}{k2}}$ $q = 2207.38$

$$p := \sqrt{\frac{2\cdot Q\cdot k0}{\tau\cdot k1}}\cdot\sqrt{\frac{k2}{k1+k2}} \qquad p = 1687.99$$

Optimal quantity of intervened acquisition /pieces/: $\Delta q := q - p$ $\Delta q = 519.38$

Minimal total costs of inventory /mu/:

$$\min(F) := \sqrt{2\cdot Q\cdot\tau\cdot k0\cdot k1}\cdot\sqrt{\frac{k2}{k1+k2}} \qquad \min(F) = 1620474.88$$

Checking of total costs for calculated optimal values of inventing /mu/:

$$minF := k0\cdot\frac{Q}{q} + k1\cdot\frac{p^2}{2\cdot q}\cdot\tau + k2\cdot\frac{(q-p)^2}{2\cdot q}\cdot\tau \qquad \text{so is:} \qquad minF = 1620474.88$$

Optimizing cycle of inventory converting /year/: $t := \frac{\tau}{Q}\cdot q$ $t = 0.108$

Cycle of regular consumption /year/: $t1 := \frac{p}{q}\cdot t$ $t1 = 0.083$

Consumption cycle of intervened quantity/year/: $t2 := t - t1$ $t2 = 0.025$

Optimal number of series per year/series/year/: $n := \frac{\tau}{t}$ $n = 110.99$

Necessary quantity in view of square equation solution per argument p. If: (nonoptimal) inventory quantity for series q and total costs F1:

$$q := 3400 \qquad F1 := 1900000$$

Square equation of costs is: $k0\cdot\frac{Q}{q1} + k1\cdot\frac{p^2}{2\cdot q1}\cdot\tau + k2\cdot\frac{(q1-p)^2}{2\cdot q1}\cdot\tau - F1 = 0$

System variable: $ORIGIN := 1$

with polynominal coefficients:

$$p12 := \begin{bmatrix} k0\cdot\frac{Q}{q} + k2\cdot\frac{q}{2}\cdot\tau - F1 \\ -k2\cdot\tau \\ \tau\cdot\left(\frac{k1+k2}{2\cdot q}\right) \end{bmatrix}$$

Possible order *p* /pieces/ makes: $\begin{pmatrix} p1 \\ p2 \end{pmatrix} := \text{polyroots}(p12)$

$$\begin{pmatrix} p1 \\ p2 \end{pmatrix} = \begin{pmatrix} 2141.8 \\ 3058.2 \end{pmatrix}$$

It follows that intervened quantity /pieces/in these conditions equal:

$$\Delta q1 := q - p1 \qquad \Delta q1 = 1258.204 \qquad\qquad \Delta q2 := q - p2 \qquad \Delta q2 = 341.796$$

cycle of regular consumption /years/: $t11 := \frac{p1}{q} \cdot t \qquad t11 = 0.068$

Consumption cycle of intervened quantity /year/:

$$t21 := t - t11 \qquad t21 = 0.04 \qquad t12 := \frac{p2}{q} \cdot t \qquad t12 = 0.097 \qquad t22 := t - t12 \qquad t22 = 0.011$$

Verification of costs solution /um/: $\begin{bmatrix} k0 \cdot \frac{Q}{q} + k1 \cdot \frac{p1^2}{2 \cdot q} \cdot \tau + k2 \cdot \frac{(q - p1)^2}{2 \cdot q} \cdot \tau \\ k0 \cdot \frac{Q}{q} + k1 \cdot \frac{p2^2}{2 \cdot q} \cdot \tau + k2 \cdot \frac{(q - p2)^2}{2 \cdot q} \cdot \tau \end{bmatrix} = \begin{pmatrix} 1900000 \\ 1900000 \end{pmatrix}$

and it full suits to previously defined costs:

13. NETWORK PLANNING

13.1 Application of network technique PERT [PN1.mcd]

Example Determine expected values of time and standard deviation for set of six activities, given through network diagram (fig.13.1). Optimistic (a), the most probable (m) and pessimistic (b) times are presented in vector way. Check probability P of critical path realization in PERT network if the planned time of its realization Tp = 12 days.

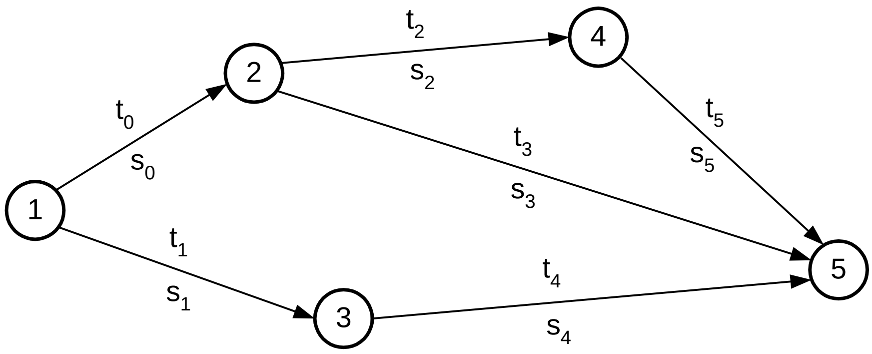


Fig. 13.1 *Network diagram PERT with marks of planning parameters*

Known time parameters: $Tp := 12$ $a := \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 6 \\ 1 \end{pmatrix}$ $m := \begin{pmatrix} 2 \\ 3 \\ 3 \\ 4 \\ 8 \\ 2 \end{pmatrix}$ $b := \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \\ 10 \\ 4 \end{pmatrix}$

Expected times *t* and adequate standard deviations *s*:

$$t := \frac{1}{6}\cdot(a + 4\cdot m + b) \qquad s := \frac{b-a}{6}$$

$$t^T \rightarrow \begin{pmatrix} 2 & \frac{17}{6} & \frac{19}{6} & 4 & 8 & \frac{13}{6} \end{pmatrix} \qquad s^T \rightarrow \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & \frac{1}{2} & \frac{2}{3} & \frac{2}{3} & \frac{1}{2} \end{pmatrix}$$

Critical (the longest expected) flow T in network and deviations σ of time activities:

$$T := \max\left(\begin{pmatrix} t_0 + t_2 + t_5 \\ t_0 + t_3 \\ t_1 + t_4 \end{pmatrix}\right) \qquad \sigma := \begin{bmatrix} \sqrt{(s_0)^2 + (s_2)^2 + (s_5)^2} \\ \sqrt{(s_0)^2 + (s_3)^2} \\ \sqrt{(s_1)^2 + (s_4)^2} \end{bmatrix} \qquad \sigma = \begin{pmatrix} 0.782 \\ 0.745 \\ 0.833 \end{pmatrix}$$

$$T = 10.833 \qquad \sigma_2 = 0.833$$

Probability P that critical flow will be realized in planned time Tp is relatively high and makes:

$$P := \mathrm{pnorm}(Tp, T, \sigma_2) \qquad P = 91.924\%$$

14. TESTING STATISTICAL HYPOTHESES

14.1 Test χ^2 for verification the hypothesis about composition of empirical with theoretical exponential distribution [ST1.mcd]

Example It is supposed that obtained numerical data adapt to exponential Puason's distribution. Graphical presentation of Puason's distribution results for parameter experiments that have been given is the file SM8.prn. the obtained empirical results of quality characteristics are grouped in tables and graphically interpreted.

Function for import of empirical data: P := READPRN("SM8.prn")

Real values vector of quality characteristic:

$P^T =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	0	4	3	2	4	10	3	6	4	5	4	4	3	7	5	5	2	3	3

π **Statistic fixing of sample**

Number of items in sample: $n := \text{length}(P)$ $n = 200$

Midvalue and extreme values are: $\text{mean}(P) = 3.975$ $\begin{pmatrix} \min(P) \\ \max(P) \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$

Interval indexes in P sample: $i := 0..199$

The number of nonstandard (spoiled) pieces in a sample: $NS := \sum P$ $NS = 795$

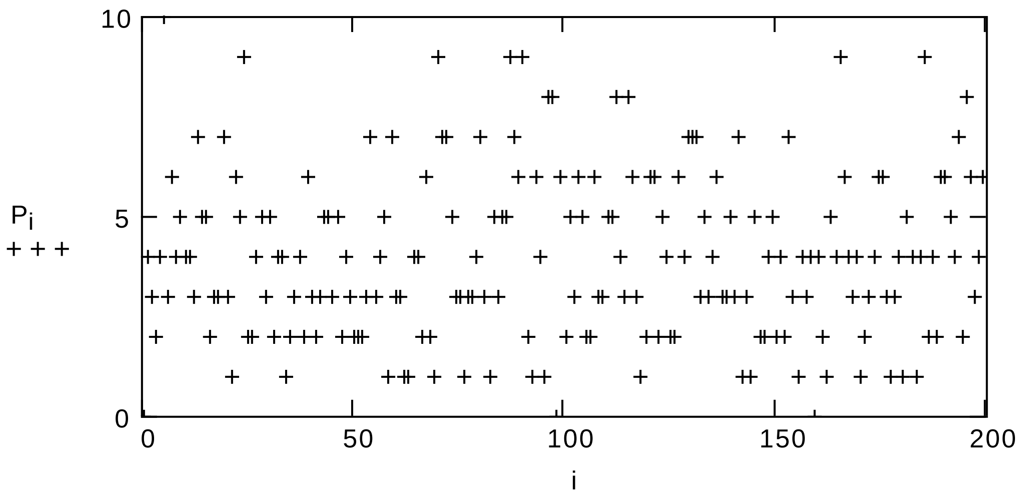


Fig. 14.1 *Value(s) arrangement of experimental data in relation to sample interval*

Number of group intervals: $h := \text{ceil}(5 \cdot \log(n))$ $h = 12$

Difference of extreme values of vector elements: $\rho := \max(P) - \min(P)$ $\rho = 10$

Integer length of group differences: $\Delta := \text{ceil}\left(\frac{\max(P) - \min(P)}{h}\right)$ $\Delta = 1$

Sample deviation: $S(P) := \text{stdev}(P)$ $S(P) = 2.103$

Sample deviation: $S(P) := \text{stdev}(P)$ $S(P) = 2.103$

Marginal values of intervals of fixed set of data: $j := 0 .. h$

Marginal values vector of group intervals: $I_j := \min(P) + \Delta \cdot j$

$I^T =$

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	1	2	3	4	5	6	7	8	9	10	11	12

Function of histograms in vector form: $f := \mathrm{hist}(I, P)$

$f^T =$

	0	1	2	3	4	5	6	7	8	9	10	11
0	3	19	31	40	34	26	21	14	5	6	1	0

Funkcija fitovane normalne raspodele: $ft(x) := n \cdot \Delta \cdot \mathrm{dpois}(x, \mathrm{mean}(P))$

Frequency of fitted data in h (of) group intervals: $k := 0 .. h - 1$
Frequencies: empirical, theoretical sum of frequencies:

$f_k =$	$ft(I_k) =$
3	3.756
19	14.93
31	29.672
40	39.316
34	39.07
26	31.061
21	20.578

$\sum f = 200$ $\sum_k ft(I_k) = 199.83$

π **Graphical interpretation of sample date distribution**

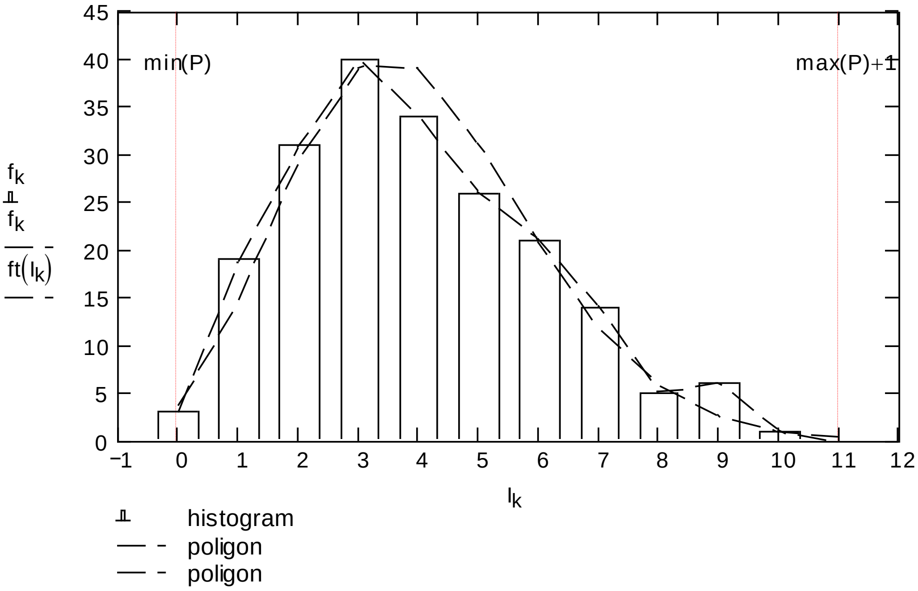
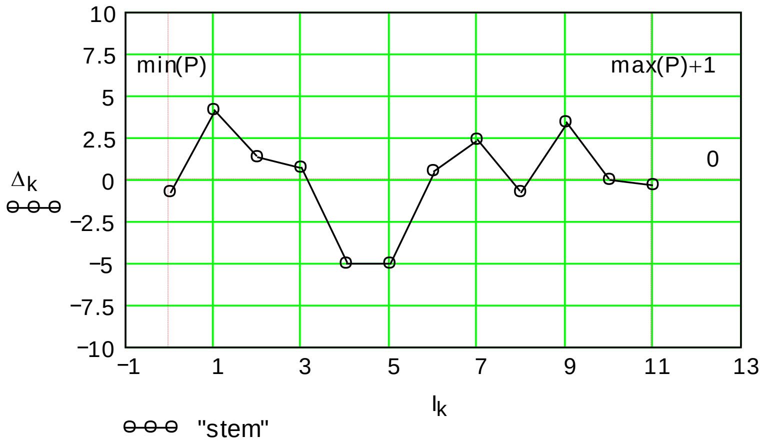


Fig. 14.2 *Histogram and polygon of empirical and polygon of theoretical distribution*

Difference function of empirical and theoretical values of distribution: $\Delta_k := f_k - ft(l_k)$



$l_k =$	$\Delta_k =$
0	-0.756
1	4.07
2	1.328
3	0.684
4	-5.07
5	-5.061
6	0.422
7	2.315
8	-0.806
9	3.436
10	-0.019
11	-0.368

Fig. 14.3 *"Stem" form of aberration function*

Note In the next procedure has been performed grouping of starting and final values of frequencies, so that the value of these additions, as new elements frequencies vectors would not be lesser (fewer) than 5. In this case (it) is reduced with 12 on 9 elements given in the next tables (following).

Distribution correction of empirical and theoretical frequencies in (m+1) interval(s):

$m := (h - 1) - 4$ $\qquad m := 8$ $\qquad g := 0 .. m$

$F_g :=$	$Ft_g :=$
$f_0 + f_1$	$ft(l_0) + ft(l_1)$
f_2	$ft(l_2)$
f_3	$ft(l_3)$
f_4	$ft(l_4)$
f_5	$ft(l_5)$
f_6	$ft(l_6)$
f_7	$ft(l_7)$
f_8	$ft(l_8)$
$f_9 + f_{10} + f_{11}$	$ft(l_9) + ft(l_{10}) + ft(l_{11})$

$g =$	$F_g =$	$Ft_g =$	$F_g - Ft_g =$	$(F_g - Ft_g)^2$
0	22	18.685	3.315	10.987
1	31	29.672	1.328	1.762
2	40	39.316	0.684	0.468
3	34	39.07	-5.07	25.708
4	26	31.061	-5.061	25.613
5	21	20.578	0.422	0.178
6	14	11.685	2.315	5.358
7	5	5.806	-0.806	0.65
8	7	3.952	3.048	9.29

π **Hypothesis testing about presumed distribution**

Testing is performed on the basis of *hi*-square test:

Empirical value of *hi*-square makes: $\chi 2 := \sum_{g} \frac{(F_g - Ft_g)^2}{Ft_g}$ $\chi 2 = 5.072$

Risk and certainty of hypothesis acceptance of (about) accordance of empirical and theoretical distribution:

$\alpha := 0.10$ $1 - \alpha = 90\%$

Number of unknown parameters of basic population (value λ): $s := 1$

Number of addends in the expression for *hi*-square value: $r := m + 1$ $r = 9$

Number of freedom degree: $\upsilon := r - s - 1$

Theoretical (limited) value hi-square: $\chi 2_g := \mathrm{qchisq}(1 - \alpha, \upsilon)$ $\chi 2_g = 12.017$

Axis scaling of hi-square frequency: $q := 0, 0.1 .. 25$ $k := 0 .. 1$

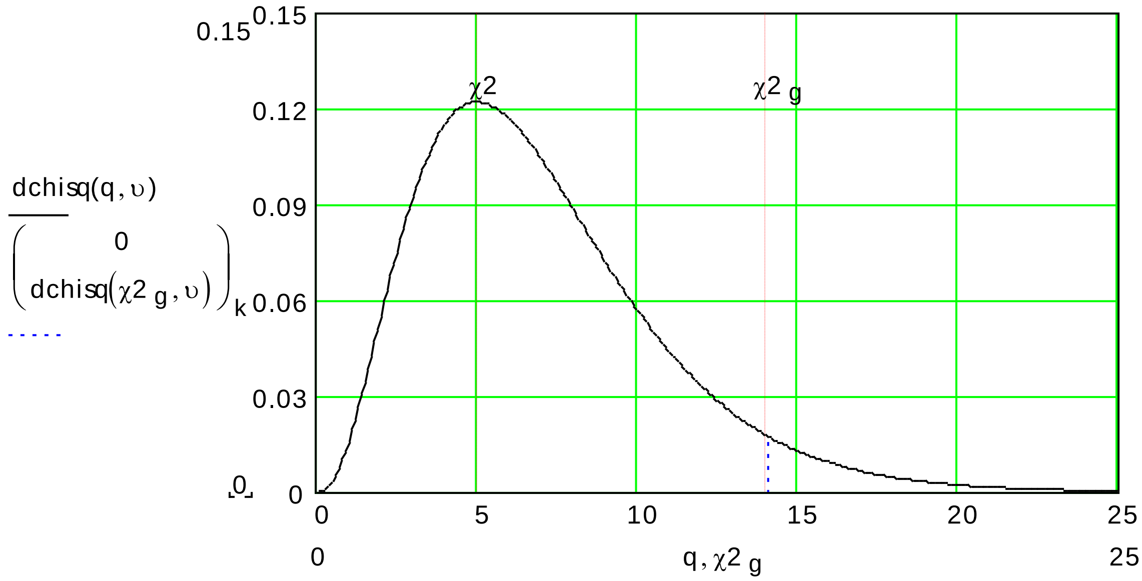


Fig. 14.4 *Graphical presentation of hypothesis testing on the basis of distribution hi-square*

π **Results of testing with Hi-square test**

Hi-square testing results may be quantatively expressed by confirming or rejecting the next hypotheses:

- Hypothesis H0: Empirical data are submitted to the theoretical low of exponential distribution.
- Hypothesis H1: Deviation empirical from theoretical distribution has no random character.

Control vector of hi-square value: $\left(0 \quad \chi 2 \quad \chi 2_{g}\right) = (0 \quad 5.072 \quad 12.017)$

Allowed hi-square values: $0 < \chi 2 < \chi 2_{g} = 1$

Area of accepting H0 hypothesis: $\mathrm{pchisq}(\chi 2, \upsilon) < 1 - \alpha = 1$

If: a) result is 1 we accept H0 hypothesis
b) result is 0 we accept H1 hypothesis.

Conclusion On the basis of confirming H0 hypothesis, we can with high probability of 90% claim that general population, from which the sample is analyzed, corresponds with supposed Puason's distribution low. Thus, hypothesis about this distribution does not contradict to empirical data.

14.2 Testing hypothesis about samples homogenity in view of their variances [ST2.mcd]

Example Perform statistical *F*-hypothesis testing about marks equality in the mathematics subject with two pupils population on the basis of variances marks. Basic function of variances analysis is to find out are there statistically significant deviations between selected sharacteristics of two groups of among many groups in population. In pedagogical sense it e.g. can be marks of pupils knowledge, time periods of education in certain areas and the like. Thereby, values, that in principle, may be qantified. If it is found that that difference between characteristics of some groups statistically significant, then it is the proof that they do not belong to the same population, and that pedagogical concluding on their baisis is unreliable. If it is found that that difference statistically insignificant (little or that is does not exist), then it is proof that there are groups, e.g. pupils, that belong to the same population and that pedagogical concluding on their basis reliable. Calculations of vareinces and all other procedures that follow are basically simple, if user has certain foreknowledge, so that he could explain inter and final results. They are established through shecked statistical procedures. Here, these procedures will be illustrated on the results of application Fisher's *F*-test, by which we can find (non) equalities of marks at one subject. In this case is chosen mathematics and the example is made on the basis of methodology elaborated in the reference [27]. Testing of differences significances among statistical values is performed comparing value pairs from two series of statistical date, through analysis of variances. Significance of differences between two groups of pupils of the same generation in Mathematics subject. Set of marks for every group has been formed in separate files: group1.prn and group2.prn. Through Mathcad they are taken over as numerical data, forming vectors of S1 and S2 marks, through an adress with articulated path. Number of pupils is not the same for every group, and for every pupil is recorded only one mark. On account of more efficient an more suitable application, separately will be interpreted and illustrated tabling of data (vector of numbers, i.e. marks) and later graphically presented *F*-test results.

π of the first group: $S1 := READPRN("gr1.prn")$

π of the second grpoup: $S2 := READPRN("gr2.prn")$

$S1^T =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	5	4	2	3	4	5	4	4	3	4	3	5	3	4	2	5	4	3	5	5	4

$S2^T =$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	4	3	1	4	5	4	5	3	4	2	4	5	3	5	5	4	4	5	2	5	4

Number of tested pupils of the first group: $n1 := length(S1)$ $n1 = 59$

Number of tested pupils of the second group: $n2 := length(S2)$ $n2 = 45$

Accepted probability of mistake of hypothesis accepting about homogenity of marks:

$$\alpha := 0.10 \qquad \alpha = 10\%$$

Accepted probability of hypothesis accepting about homogenity of marks:

$$1 - \alpha = 0.9 \qquad 1 - \alpha = 90\%$$

Standard deviation and variance of samples:

prve grupe: $stdev(S1) = 0.9579$ $var(S1) = 0.918$

druge grupe: $stdev(S2) = 1.0376$ $var(S2) = 1.077$

Objective mark of samples variance:

of the first group: $v1 := \frac{n1}{n1 - 1} \cdot var(S1)$ $v1 = 0.9334$

of the second group: $v2 := \frac{n2}{n2 - 1} \cdot var(S2)$ $v2 = 1.101$

Minimal, average and i maximal mark of samples:

of the first group: $min(S1) = 2$ $mean(S1) = 3.78$ $max(S1) = 5$

of the second group: $min(S2) = 1$ $mean(S2) = 3.889$ $max(S2) = 5$

π ***F*–hypothesis testing of pupils' marks equality**

Number of freedom degrees:

of the first group: $\upsilon1 := n1 - 1$ $\upsilon1 = 58$

of the second group: $\upsilon2 := n2 - 1$ $\upsilon2 = 44$

Upper threshold of significance: $F1 := qF\left(\frac{\alpha}{2}, \upsilon1, \upsilon2\right)$ $F1 = 0.6308$

Lower threshold of significance: $F2 := qF\left(1 - \frac{\alpha}{2}, \upsilon1, \upsilon2\right)$ $F2 = 1.6134$

Variable of F distribution: $F := \mathrm{if}\left(\frac{v1}{v2} > 1, \frac{v1}{v2}, \frac{v2}{v1}\right)$ $F = 1.1796$

Note The variable *F* is necessary to nominate to be greater than 1. In that sense *if* function is used.

Scaled values of independent variable: $q := 0, 0.02 .. 4$ $k := 0 .. 1$

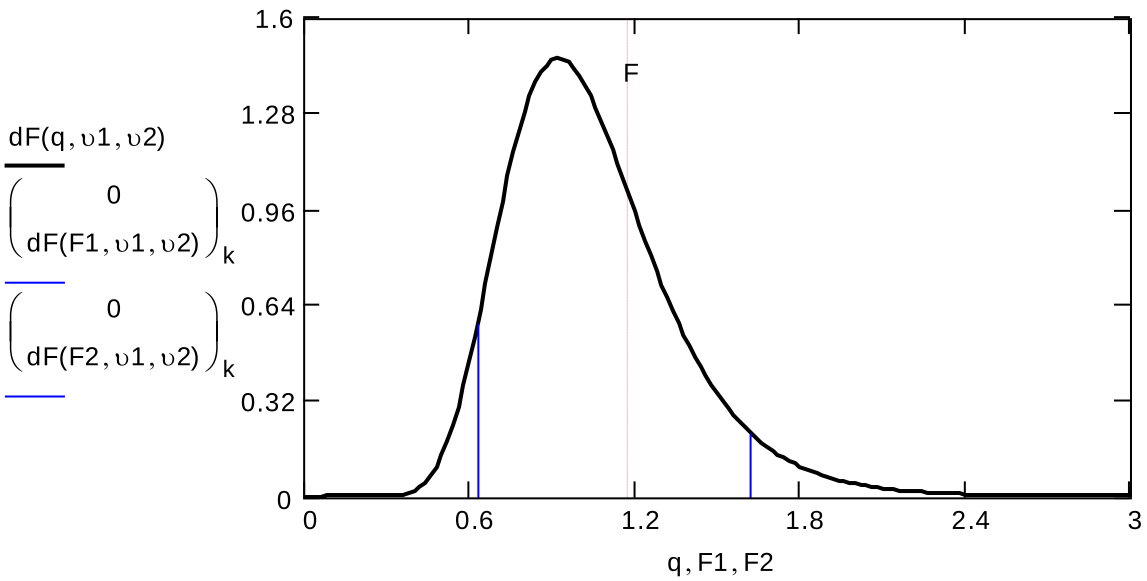


Fig. 14.5 *Graphical presentation of hypothesis testing on the basis of F-distribution*

Control vector of limit *F* values: $(F1 \quad F \quad F2) = (0.6308 \quad 1.1796 \quad 1.6134)$

π Hypothesis H0: $\sigma_{01}^{\ 2} = \sigma_{02}^{\ 2}$ that variances of basic sets are nutually equal.

π Hypothesis H1: $\sigma_{01}^{2} \neq \sigma_{02}^{2}$ that among variances of basic sets (there) are significant differences.

Interval of confidence: $\frac{\alpha}{2} < pF(F, \upsilon 1, \upsilon 2) < 1 - \frac{\alpha}{2} = 1$

Area of H0 hypothesis accepting: $F1 < F < F2 = 1$

Test criterion: If: *a)* result is 1 H0 hypothesis is accepted

b) result is 0 H1 hypothesis is accepted.

Conclusion Hypothesis H0 is confirmed. In that case it can be stated that estimates of variances are not significantly differetn, so there are no bases for suppostition that one group of pupils has achieved better results in mathematics than the other one. Here presented method of pedagogical research may be base for development of extended model of pedagogical experiment on the basis of new and more detailed data.

Reference

[1] Backović, M. i Vuleta, J. EKONOMSKO MATEMATIČKI METODI I MODELI, Ekonomski fakultet, Beograd, 2004.

[2] Backović, M., Vuleta, J., Prica, I. i Popović, Z. EKONOMSKO MATEMATIČKI METODI I MODELI, Zbirka rešenih problema, Ekonomski fakultet, Beograd, 2004.

[3] Baets, B., Fodor, J. i Radojević, D. (Editors). FUZZY FOR BETTER, Eurofuse Anniversary Workshop, Institut “Mihajlo Pupin”, Belgrade, 2005.

[4] Burton, V. D. OPERACIONA ISTRAŽIVANJA U ISTRAŽIVANJIMA I RAZVOJU, (prevod Zbornika radova), “Savremena administracija”, Beograd, 1968.

[5] Cvetković, D., Kovačević-Vujčić, V. (redaktori) KOMBINATORNA OPTIMIZACIJA, Matematička teorija i algoritmi, “DOPIS”, Beograd, 1996.

[6] Ćerić, V. METODE VREDNOVANJA SIMULACIONIH MODELA, SYM-OP-IS ‘84., (str. 169-177), Herceg Novi, 1984.

[7] Čileg, M., Vugdelija, D. i Kiš, T. OPERACIONA ISTRAŽIVANJA, Zbirka zadataka sa rešenjima, Ekonomski fakultet, Subotica, 1996.

[8] Čupić, M., Suknović, M., Radojević, G. i Jovanović, V. SPECIJALNA POGLAVLJA IZ TEORIJA ODLUČIVANJA: KVANTITATIVNA ANALIZA, Univerzitet u Novom Sadu, Fakultet tehničkih nauka, Novi Sad, 2004.

[9] Dantzig, G, B. LINEAR PROGRAMMING AND EXTENSIONS, Princeton University Press, Princeton, New Jersey, 1963.

[10] Fishman, S. G. MONTE CARLO CONCEPTS, ALGORITHMS AND APPLICATIONS, Springer Series in Operations Research, Springer, 1999.

[11] Ivanović, G. i Stanivuković, D. POUZDANOST TEHNIČKIH SISTEMA - Zbirka rešenih zadataka, Univerzitet u Beogradu, Mašinski fakultet, Beograd, 1987.

[12] Krčevinac, S., Čangalović, M., Kovačević-Vujić, V., Martić, M. i Vujošević, M. OPERACIONA ISTRAŽIVANJA, Fakultet organizacionih nauka, Beograd, 2004.

[13] Kun, L. PRIMENA ISTRAŽIVANJA OPERACIJA, Mašinski fakultet, Novi Sad, 1973.
[14] Letić, D., Davidović, B., Berković, I. i Petrov, T. MATHCAD 13 U MATEMATICI I VIZUELIZACIJI, Kompjuter biblioteka, Čačak, 2007.
[15] Letić, D.i Jevtić, V. OPERACIONA ISTRAŽIVANJA, Tehnički fakultet "M. Pupin", Zrenjanin, 2006.
[16] Marinković, M. OPERACIONA ISTRAŽIVANJA, Viša škola za organizaciju rada, Novi Sad, 1987.
[17] Nikolić, VIŠEKRITERIJUMSKA OPTIMIZACIJA – METODE, LOGISTIKA, SOFTVER, Visoka vojna škola Vojske Jugoslavije, Beograd, 1999.
[18] Opricović, S. OPTIMIZACIJA SISTEMA – ZADACI IZ VODOPRIVREDE I HIDROTEHNIKE, Građevinski fakultet, Beograd, 1995.
[19] Petrić, J. OPERACIONA ISTRAŽIVANJA I i II, Naučna knjiga, Beograd, 1989.
[20] Petrić, J., Šarenac, L. i Kojić, Z. OPERACIONA ISTRAŽIVANJA I i II, Zbirke rešenih zadataka, Naučna knjiga, Beograd, 1992.
[21] Саздановић, С. ЛИНЕАРНО ПРОГРАМИРАЊЕ, Научна књига, Београд, 1988.
[22] Stanojević, R. METODA SIMPLEKS, ZBIRKA REŠENIH PRIMERA, Institut "Braća Karić", Beograd, 1996.
[23] Stojanović, D. EKONOMSKO MATEMATIČKI METODI I MODELI, Ekonomski fakultet, Beograd, 1990.
[24] Vujanović, N. TEORIJA POUZDANOSTI TEHNIČKIH SISTEMA, Vojnoizdavački i novinski centar, Beograd, 1987.
[25] Vujošević, M. OPERACIONA ISTRAŽIVANJA, Izabrana poglavlja, Fakultet organizacionih nauka, Beograd, 1999.
[26] Vukadinović, S. MASOVNO OPSLUŽIVANJE, Naučna knjiga, Beograd, 1986.
[27] Vukadinović, S. ELEMENTI TEORIJE VEROVATNOĆE I MATEMATIČKE STATISTIKE, Privredni pregled, Beograd, 1981.
[28] Вукадиновић, С. Цвејић, С. МАТЕМАТИЧКО ПРОГРАМИРАЊЕ, Универзитет у Приштини, Приштина, 1996.
[29] Vukadinović, S. i Popović, J. METODE MONTE KARLO, Saobraćajni fakultet, Beograd, 1992.
[30] Vukičević, R. M. OPERACIONA ISTRAŽIVANJA u preradi drveta, Univerzitet u Novom Sadu, Šumarski fakultet, Beograd, 1997.
[31] Winston, L.W. OPERATIONS RESEARCH, Applications and Algorithms, Indiana University, Duxbury Press, Belmont, California, 1994.
[32] Zelenović, D., Ćosić, I. i Maksimović, R. PROJEKTOVANJE PROIZVODNIH SISTEMA, Priručnik za vežbe, Nauka, Beograd, 1992.
[33] Zlobec, S., Petrić, J. NELINEARNO PROGRAMIRANJE, Naučna knjiga, Beograd, 1989.

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